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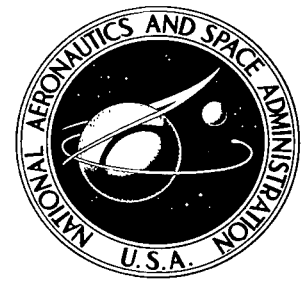

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APPLICATION OF THE CHARACTERISTICS METHOD TO NUMERICAL SOLUTION OF UNIDIMENSIONAL PROBLEMS IN GAS DYNAMICS

by A. I. Zhukov

Izdatel'stvo Akademii Nauk SSSR,
Moscow, 1960

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By A. I. Zhukov

Translation of "Primeneniye metoda kharakteristik k chislennomu resheniyu odnomernykh zadach gasovoy dinamiki. Trudy Matematicheskogo Instituta imeni V. A. Steklova, 58." Izdatel'stvo Akademii Nauk SSSR, pp. 1-149, Moscow, 1960.

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TRANSACTIONS OF THE V. A. STEKLOV MATHEMATICAL INSTITUTE

APPLICATION OF THE CHARACTERISTICS METHOD TO NUMERICAL SOLUTION OF UNIDIMENSIONAL PROBLEMS IN GAS DYNAMICS

A.I. Zhukov

ABSTRACT

The motion of an ideal, compressible gas is investigated, assuming that all quantities depend on one coordinate and time. The usual gasdynamic equations are derived in Chapter 1.

In Chapter 2, the equations of the characteristics and the invariants are discussed. Chapter 3 treats the theory underlying the practical methods used to determine the flow field. Chapter 4 outlines the order of accuracy of the initial calculation (based on linear interpolation) and the recalculation (based on quadratic interpolation). Practical formulas are derived for several equations of state in Chapter 5, and practical procedures for hand computation are discussed in Chapter 6.

The computation of the flow field in the vicinity of the boundary is treated in Chapter 7. Several examples of simple waves (for which one of the Riemann invariants is constant) are discussed. Using the laws of conservation, the jumps in the flow parameters across line discontinuities are presented in Chapter 9.

Discontinuities through which no matter passes are called contact discontinuities. If there is a flux of matter through the discontinuity, it is called a shock wave. Practical formulas for computing the flow field with a contact discontinuity are discussed in Chapter 10. The decay of an arbitrary discontinuity into either a shock wave, expansion wave, or contact discontinuity is treated.

Under the assumption that the substance cannot sustain negative pressure, separation occurs which is discussed in Chapter 14. The over-all accuracy of the solution is checked by verifying the fact that the integral forms of the conservation laws are satisfied. A detailed description of the computer program for the "Strela" computer and an example are given.

FORWARD

/4*

By right, the method of characteristics occupies an important position in hydrodynamics and gas dynamics. However, there has been no complete description of it in our literature as a numerical method until recently. The book by D. Yu. Panov* gives only the simplest information regarding this problem. It cannot serve as a handbook for the practical solution of any complex problem.

This book represents an attempt to present systematically a numerical method of characteristics as applied to one specific class of problems - unidimensional, nonstationary problems of gas dynamics. It includes basic information on equations of hydrodynamics which, however, can in no way substitute for a systematic course. The book is designed for the reader who has a basic understanding of the theory of equations of partial derivatives, who is familiar with the bases of hydrodynamics and gas dynamics, and who has studied numerical methods in the normal university course. Appendix 1 describes programs designed for electronic computers; this section is intended for qualified programmers.

The bases of the numerical method presented here were developed more than ten years ago by K. A. Semendyayev with the help of I. M. Gel'fand. The individual sections belong to the author of this book.

The manuscript of this book was reviewed, and the author would like to express his profound appreciation to K. A. Semendyayev, S. K. Godunov, and Ya. M. Kazhdan who made several very valuable comments.

---

* D. Yu. Panov. Numerical Solution of Quasilinear Hyperbolic Systems of Differential Equations of Partial Derivatives (Chislennoye resheniye kvazilineynykh giperbolicheskikh sistem differentsial'nykh uravneniy v chastnykh proizvodnykh). Moscow, Gostekhizdat, 1957.

* Note: Numbers in the margin indicate pagination in the original foreign text.

1. BASIC EQUATIONS

We shall investigate the problem of the unidimensional motion of an /5
ideal compressible liquid (or gas). As is known, the motion of a solid medium is called unidimensional, if the spatial distribution of all the quantities characterizing its condition depends only on one coordinate at each moment in time. This coordinate may be the customary Cartesian coordinate; such motion is called flat unidimensional motion. This coordinate may also be the distance to a certain fixed axis of symmetry, and then the motion will be cylindrically symmetrical motion. Finally, one significant coordinate may be the distance to a specific center - this motion is called spherically symmetrical motion. We shall now derive the main equations for unidimensional gas motion.

A typical example of flat unidimensional motion is the motion of a gas or liquid within the length of a right cylindrical tube (Figure 1). It must thus be assumed that the tube walls have no influence upon the motion of the substance within, and that all the motion characteristics (particle velocity, density, pressure, etc.) are the same for any transverse cross section and for all points of this cross section. Particle velocities must be parallel to the tube axis.

By selecting one specific cross section as the origin, we can completely define any other cross section by the single coordinate x. Any characteristic of the moving medium will depend only on this coordinate x and the time t.

In order to simplify the later discussion, we shall assume that the area of the tube transverse cross section equals unity (this does not impose a limitation on the generality, since - due to the fact that the walls have no influence - the motion of the medium does not depend on the magnitude of the transverse cross section). In addition, we shall assume that the total mass /6
of a substance located to the left of any cross section x, at any moment in time t, has a specific finite value M. The quantity M is a function of the two variables x and t. Let us determine its total differential.

Let us first examine the condition of our substance at a certain fixed moment in time t. If x and x+dx are two infinitely close cross sections, then the mass of the substance between them will be

$$dM = \rho\, dx,$$

where ρ is the density at the point x under consideration. Consequently, we have

$$\frac{\partial M}{\partial x} = \rho.$$

On the other hand, let us examine a fixed cross section x, and let us calculate the change in the mass M during the time dt. The volume of substance passing through our cross section will be u dt, where u is the particle velocity. We obtain the mass by multiplying this volume by the density. We thus have

$$dM = -\rho u\, dt.$$

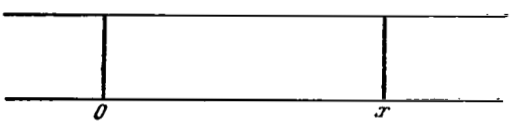

Figure 1

The minus sign must be placed here, because at a positive velocity u the particles move from left to right, and consequently the mass M decreases. This means that

$$\frac{\partial M}{\partial t} = -\rho u.$$

We finally have

$$dM = \rho\, dx - \rho u\, dt. \tag{1.1}$$

Let us now assume that our cross section, defined by the coordinate x, moves according to a certain law x = x(t). This motion can be expressed graphically on the x, t plane (Figure 2). We can calculate the mass flux through this cross section when it moves from point A to point B. To do this, it is necessary to integrate the differential (1.1) along the curve AB, i.e., to calculate the integral

$$\int_A^B \rho\, dx - \rho u\, dt.$$

It can be readily shown that the magnitude of this integral may depend (for a given motion of the substance) only on the points A and B, but not on the integration path combining them (formally, this arises from the fact that expression (1.1) is a total differential). In particular, integration of expression (1.1) along any closed contour on the x, t plane must yield zero as a result:

$$\oint \rho\, dx - \rho u\, dt = 0. \tag{1.2}$$

This formula represents the most general mathematical expression for the /7 physical law of conservation of mass for the case under consideration of the flat unidimensional motion of a solid medium. It must be noted that equation (1.2) is valid for any integrable functions $\rho(x, t)$ and $u(x, t)$. In particular, these functions may be discontinuous.

If the functions of ρ and u have continuous partial derivatives, then condition (1.2) can be written in the form

$$\frac{\partial \rho}{\partial t} = -\frac{\partial (\rho u)}{\partial x},$$

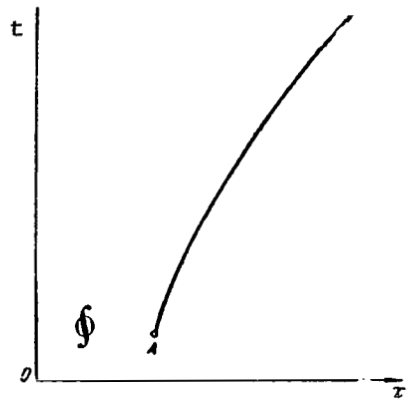

Figure 2

or

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0. \qquad (1.3)$$

This is a differential form of the law of conservation of mass. As is known, equation (1.3) is called the equation of discontinuity.

Similar expressions for cylindrically symmetric, and for spherically symmetric, motion may be readily obtained. To be specific, let us discuss the spherical case. All of the physical characteristics of the medium -- density, pressure, absolute particle velocity, etc. - at any moment in time will be constant on the surface of any sphere drawn around the center of symmetry. The direction of velocity at each point coincides with the direction of the radius vector. Therefore, the velocity is given by one scalar quantity $u = \frac{dr}{dt}$.

Let us draw a sphere having the radius r around the center of symmetry, and let us use $4\pi M$ to designate the total mass of the substance included within it (for purposes of convenience, the factor 4π is introduced). The volume of substance included between the spheres having the radius r and r + dr equals $4\pi r^2$ dr, and its mass is $4\pi\rho r^2$ dr. Therefore, we have

$$\frac{\partial M}{\partial r} = \rho r^2.$$

The volume of substance passing through a sphere having the radius r during the time dt equals the product of the surface of this sphere $4\pi r^2$ by the quantity u dt. Thus, multiplying by the density, we obtain

$$\frac{\partial M}{\partial t} = -u\rho r^2$$

(the minus sign is used for the same reasons given above). Consequently, /8 we have

$$dM = \rho r^2 dr - \rho u r^2 dt.$$

We thus rapidly obtain the integral form of the law of conservation of mass

$$\oint \rho r^2 dr - \rho u r^2 dt = 0$$

and its differential form

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} = -2\frac{\rho u}{r}$$

(spherical equation of discontinuity).

In the cylindrical case, similar reasoning leads to the expression for the mass differential

$$dM = \rho r\, dr - \rho u r\, dt$$

(r - distance from axis of symmetry, u - radial velocity component). We thus have

$$\oint \rho r\, dr - \rho u r\, dt = 0,$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} = -\frac{\rho u}{r}.$$

All three cases can be combined, if it is stipulated first of all that the spatial coordinate is designated by r also for flat unidimensional motion, and secondly that the factor ν is introduced, equalling 0 in the flat case, 1 - in the cylindrical case, and 2 - in the spherical case. The mass differential can be written as follows

$$dM = \rho r^\nu dr - \rho u r^\nu dt, \tag{1.4}$$

the law of conservation of mass - in the following form

$$\oint \rho r^\nu dr - \rho u r^\nu dt = 0, \tag{1.5}$$

and the equation of discontinuity

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} = -\nu\frac{\rho u}{r}. \tag{1.6}$$

In order to avoid any confusion, it should be noted that M, which can be determined by equation (1.4), has the dimension $[ML^{\nu-2}]$, which coincides with the dimension of mass only in the spherical case. The fact is that in the flat problem M represents the mass of the substance per unit of transverse tube cross section, and in the cylindrical problem it represents the mass per unit of length of the axis of symmetry. A similar stipulation must be kept /9 in mind for the momentum and energy examined below.

Let us investigate the law of conservation of momentum. The condition of ideality for the substance is important for its derivation, i.e., the absence of internal friction in it. Therefore, the only interaction force between the particles will be the force of the pressure p.

In the case of flat, unidimensional motion, no particular difficulties are encountered in deriving the law of conservation of momentum. We shall again employ our model of a gas or liquid within a tube, and we shall designate the total momentum of the substance located to the left of the cross section with the coordinate r by P. We shall assume from this point on that not only is this momentum P finite, but that there is no momentum flux from minus infinity, i.e., for example, somewhere on the left there is a region of zero pressure (just as previously, the final results do not depend on these assumptions).

The mass $\rho\, dr$ is included between the cross sections r and r + dr. In order to obtain the momentum, it is necessary to multiply it by the velocity u. Thus, we have

$$\frac{\partial P}{\partial r} = \rho u.$$

The mass $\rho u\, dt$ passes through the given (stationary) transverse cross section in the time dt; the momentum $\rho u^2 dt$ is removed together with it. In addition, the force p influences the substance with the momentum P from the side of the substance located to the right (we should recall that the transverse cross section has unit area). Therefore, the momentum p dt is lost in addition. Finally, we have

$$\frac{\partial P}{\partial t} = -(\rho u^2 + p).$$

Consequently, we have

$$dP = \rho u\, dr - (\rho u^2 + p)\, dt. \tag{1.7}$$

Both the integral and the differential form of the law of conservation of momentum can thus be readily obtained. We prefer, however, to do this somewhat later in a more general form.

The cylindrical and spherical cases are more complex. Momentum is a vectorial quantity, and it can be readily shown that, for example, in the spherical case the total momentum of a substance located within any sphere, which is drawn around the center of symmetry, always equals zero exactly. The law of

conservation of momentum becomes trivial and meaningless.

The way out of this difficulty is as follows. Let us cut a cone out of space (in the spherical case) with the apex at the center of symmetry (Figure 3), and we shall investigate only the momentum of the substance included within this cone and limited on the outside by a sphere having the radius r. It can be /10 assumed that this cone is not "imaginary", but is "real". It is only important that its walls be absolutely smooth and that they have no influence on the motion of the substance included within them (similarly to the tube walls in the flat case).

The total momentum of the substance within such a cone will not be equal to zero, generally speaking, and we may attempt to calculate its total differential. However, one feature must be taken into account here. No matter how "smooth" the walls of our cone are, they produce a pressure on the substance included within them; in contrast to the flat case, the resultant force of this pressure does not equal zero (it can be readily seen that it will be directed to the outside). We must take this additional force into account. It can be readily shown that the pressure (with the appropriate weight) must be integrated over the cone surface in order to compute this force. The momentum differential cannot be expressed only by local quantities; it contains an integral. The situation is completely identical in the cylindrical case.

We shall not perform the complete derivation here, but shall present the result at once. The momentum differential has the following form

$$dP = \rho u r^{\nu} dr - [(\rho u^2 + p) r^{\nu} - \int_0^r \nu r^{\nu-1} p \, dr] dt. \tag{1.8}$$

The following form in which it may be written is also of interest:

$$dP = \rho u r^{\nu} dr - [\rho u^2 r^{\nu} + \int_0^r r^{\nu} dp] dt. \tag{1.9}$$

and the integral here must be regarded as the Stieltjes integral.

In order to obtain the integral form for the law of conservation of momentum, we must integrate the differential (1.8) over (an arbitrary) closed contour on the r, t plane. Thus, the integral in the second component can be /11 readily transformed into an integral over the area included within this contour. Thus, the law of conservation assumes the following form (we shall again omit the intermediate computations)

$$\oint \rho u r^{\nu} dr - (\rho u^2 + p) r^{\nu} dt = \int \nu r^{\nu-1} p \, dr \, dt. \tag{1.10}$$

The integral for an arbitrary closed contour is on the left; the integration direction is assumed to be counter-clockwise (Figure 4). The integral over the area included within this contour is on the right.

By now equating the partial derivative for t from the first component of

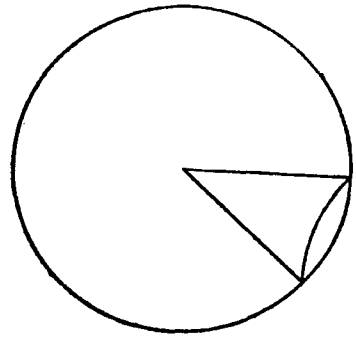

Figure 3

the right part of (1.8) with the partial derivative for r from its second component, we obtain the differential form of the law of conservation of momentum. The integral in the second component thus disappears, and we obtain the equation

$$\frac{\partial(\rho u r^{\nu})}{\partial t} = -\frac{\partial[(\rho u^2 + p) r^{\nu}]}{\partial r} + \nu r^{\nu-1} p.$$

This equation can be transformed to the following form:

$$r^{\nu} u\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \nu\frac{\rho u}{r}\right) + r^{\nu}\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r}\right) = 0.$$

When making a comparison with the equation of discontinuity (1.6), we can see that the first bracket vanishes; we then arrive at the following equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \qquad (1.11)$$

which is well known as the Euler unidimensional equation. We would like to point out that it has absolutely the same form in the flat case, as it does in the cylindrical and spherical cases.

We must now examine the law of conservation of energy. The difficulties encountered in deriving the law of conservation of momentum disappear here, because energy is a scalar quantity. It is only necessary to include again the condition of an ideal medium, which is expressed in the absence of thermal conductivity this time. This corroborates the fact that the total energy of a substance included within a certain volume changes only due to the pressure force at the boundary. Let us perform a derivation for the cylindrical case. We /12 shall draw two planes perpendicularly to the axis of symmetry; these planes are located a unit of length from each other. We shall use $2\pi E$ to designate the total energy of a substance included within a volume defined by these planes and by a cylindrical surface having the radius r which is drawn around the axis of symmetry (Figure 5).

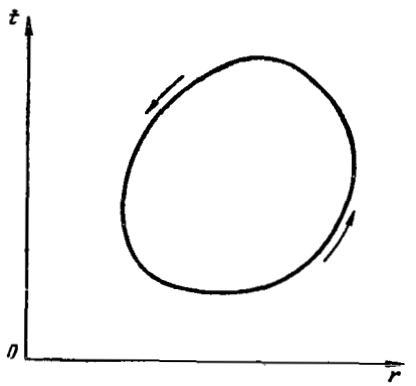

Figure 4

If we change the cylinder radius by the quantity dr, then the mass of a substance included within the cylinder changes by the quantity $2\pi\rho r\,dr$. The total energy of this substance is comprised of two parts - a kinetic and internal (thermodynamic) part. If we use ε to designate the specific internal energy (i.e., the energy of a unit of mass), we find that the total energy of a unit of mass equals $\varepsilon + \frac{u^2}{2}$. Consequently, the change in the total energy of our substance, which occurs due to a change in the cylinder radius, equals $2\pi\rho r\left(\varepsilon + \frac{u^2}{2}\right) dr$. Therefore,

$$\frac{\partial E}{\partial r} = \rho\left(\varepsilon + \frac{u^2}{2}\right) r.$$

Particles occupying the cylindrical surface move away from it at the distance $dr = u\,dt$ in the time dt. There is an outflow of energy $2\pi\rho\left(\varepsilon + \frac{u^2}{2}\right) ru\,dt$ from our cylinder along with the particles. In addition, these particles perform the work $2\pi rpu\,dt$ (the flat side of the cylinder equals $2\pi r$). The total energy decrease is $2\pi r\left[\rho\left(\varepsilon + \frac{u^2}{2}\right) + p\right] u\,dt$. Thus,

$$\frac{\partial E}{\partial t} = -u\left[\rho\left(\varepsilon + \frac{u^2}{2}\right) + p\right] r.$$

We arrive at the following expression for the energy differential:

$$dE = \rho\left(\varepsilon + \frac{u^2}{2}\right) r\,dr - u\left[\rho\left(\varepsilon + \frac{u^2}{2}\right) + p\right] r\,dt.$$

In the general case (for all three types of unidimensional motion), it has the form

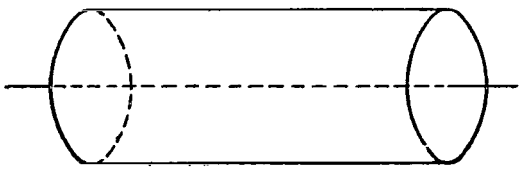

Figure 5

$$dE = \rho\left(\varepsilon + \frac{u^2}{2}\right) r^\nu dr - u\left[\rho\left(\varepsilon + \frac{u^2}{2}\right) + p\right] r^\nu dt. \tag{1.12}$$

We thus rapidly obtain the integral form for the law of conservation of energy:

$$\oint \rho\left(\varepsilon + \frac{u^2}{2}\right) r^\nu dr - u\left[\rho\left(\varepsilon + \frac{u^2}{2}\right) + p\right] r^\nu dt = 0. \tag{1.13}$$

The differential form of the law of conservation of energy can be written as /13 follows

$$\frac{\partial}{\partial t}\left[\rho\left(\varepsilon + \frac{u^2}{2}\right) r^\nu\right] + \frac{\partial}{\partial r}\left\{u\left[\rho\left(\varepsilon + \frac{u^2}{2}\right) + p\right] r^\nu\right\} = 0.$$

The latter differential equation can be greatly simplified, if we employ the thermodynamic equation

$$d\varepsilon = Tds - pd\left(\frac{1}{\rho}\right),$$

where T is absolute temperature, and s is entropy. Removing all of the brackets in this differential equation, making the following substitutions

$$\frac{\partial \varepsilon}{\partial t} = T\frac{\partial s}{\partial t} + \frac{p}{\rho^2}\frac{\partial \rho}{\partial t},$$

$$\frac{\partial \varepsilon}{\partial r} = T\frac{\partial s}{\partial r} + \frac{p}{\rho^2}\frac{\partial \rho}{\partial r},$$

and taking the equation of discontinuity (1.6) into account as well as the Euler equation (1.11), we obtain the following simple differential equation:

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial r} = 0. \tag{1.14}$$

Attention should be called to the fact that the left part of this equation represents the derivative of $\frac{ds}{dt}$ taken in the direction dr = u dt. However, the latter relationship is the law of particle motion for our substance. Consequently, equation (1.14) confirms the fact that the entropy of each particle remains constant. We should point out that this conclusion is valid only when the

differential laws of conservation are applicable, i.e., where the quantities u, ρ, p, ε, etc. are at least continuous. It may then be assumed that the thermodynamic process which each particle undergoes is reversible. The absence of thermoconductivity makes it adiabatic; therefore, equation (1.14) should be written at once. It must be pointed out that the thermodynamic equation given above is applicable only when the process is reversible.

Not only the entropy s is constant along the direction dr = u dt. Expression (1.4) shows that the quantity M does not change in this direction. In its turn, this indicates that the quantities M and s are related by the functional dependence

$$s = s(M).$$

This fact will play a very important role later on. However, it is advantageous to select another quantity having the dimensionality of length, instead of M. It is called the Lagrangian coordinate.

Let $\rho_0 = \rho_0(M)$ be the arbitrary function of M having the dimensionality of density. We shall introduce the Lagrangian coordinate R by means of the following relationship /14

$$R^{\nu} dR = \frac{dM}{\rho_0}. \tag{1.15}$$

Thus, for the given function $\rho_0(M)$ the quantity R is determined within an accuracy of the constant:

$$R^{\nu+1} = R_0^{\nu+1} + \int_0^{(r,t)} \frac{\nu+1}{\rho_0} dM.$$

Integration may be performed in any manner.

The quantity R is a function of M. Consequently, we have

$$s = s(R). \tag{1.16}$$

Substituting dM from (1.4) in (1.15), we obtain the differential of the Lagrangian coordinate

$$R^{\nu} dR = \frac{\rho}{\rho_0} r^{\nu} (dr - u\,dt). \tag{1.17}$$

Let us combine all of the main equations obtained in the present section.

The expressions for the total differentials of mass, momentum, and energy [see (1.4), (1.8), (1.12)] are:

$$\left.\begin{aligned} dM &= \rho r^{\nu} dr - \rho u r^{\nu} dt, \\ dP &= \rho u r^{\nu} dr - \left[(\rho u^2 + p) r^{\nu} - \int_0^r \nu r^{\nu-1} p\, dr\right] dt, \\ dE &= \rho \left(\varepsilon + \frac{u^2}{2}\right) r^{\nu} dr - u\left[\rho\left(\varepsilon + \frac{u^2}{2}\right) + p\right] r^{\nu} dt. \end{aligned}\right\} \tag{1.18}$$

The integral laws of conservation [see (1.5), (1.10), (1.13)]:

$$\left.\begin{aligned} \oint \rho r^{\nu} dr - \rho u r^{\nu} dt &= 0, \\ \oint \rho u r^{\nu} dr - (\rho u^2 + p) r^{\nu} dt &= \int \nu r^{\nu-1} p\, dr\, dt, \\ \oint \rho \left(\varepsilon + \frac{u^2}{2}\right) r^{\nu} dr - u\left[\rho\left(\varepsilon + \frac{u^2}{2}\right) + p\right] r^{\nu} dt &= 0 \end{aligned}\right\} \tag{1.19}$$

The differential equations of motion [see (1.6), (1.11), (1.14)]:

$$\left.\begin{aligned} \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} &= -\nu \frac{\rho u}{r}, \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} &= 0, \\ \frac{\partial s}{\partial t} + u \frac{\partial s}{\partial r} &= 0. \end{aligned}\right\} \tag{1.20}$$

We should note that the differentials (1.18) and the laws of conservation /15
(1.19) will appear simpler if they are written in Lagrangian coordinates. Actually, we have the following from (1.17)

$$r^{\nu} dr = \frac{\rho_0}{\rho} R^{\nu} dR + r^{\nu} u\, dt. \tag{1.21}$$

Substituting this in (1.18), we obtain

$$\left.\begin{aligned} dM &= \rho_0 R^{\nu} dR, \\ dP &= \rho_0 u R^{\nu} dR - \left(p r^{\nu} - \int_0^r \nu r^{\nu-1} p\, dr\right) dt, \\ dE &= \rho_0 \left(\varepsilon + \frac{u^2}{2}\right) R^{\nu} dR - u p r^{\nu} dt. \end{aligned}\right\} \tag{1.22}$$

We can transform equations (1.19) in a similar manner. It is convenient to use these relationships for transforming the differential equations (1.20) to Lagrangian coordinates.

In conclusion, we would like to point out the following. In the classical theory of differential equations, the Cauchy problem is raised for differential equations; equations (1.20) represent these equations in our case. However, as the derivation of these equations shows, the integral equations (1.19) directly expressing the physical laws of conservation must be regarded as basic for our problem. We must formulate the Cauchy problem for them, and must solve the problems immediately arising regarding the existence and uniqueness of a

solution, etc. This point of view has been developed only in very recent years, and no comprehensive results have yet been obtained.

In particular, it is clear that integral equations such as (1.19) still do not provide for a unique solution; additional conditions are requisite. In our case, such a condition is provided by the second law of thermodynamics, from which it follows that the entropy of each particle cannot decrease. Under this condition, S. K. Godunov* was able to illustrate the uniqueness of the solution for the Cauchy problem for equations (1.19), under certain limitations.

As a rule, we shall employ differential equations of motion, including integral equations, only in those cases when the differential equations are untenable (for example, when investigating and calculating discontinuous solutions).

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* S. K. Godunov. The Uniqueness of a Solution for Hydrodynamic Equations (O yedinstvennosti resheniya uravneniy gidrodinamiki). Matematicheskiy Sbornik Akademii Nauk SSSR i Moskovskogo Matematicheskogo Obshchestva, 40 (82), No. 4, 467-478, 1956.

2. CHARACTERISTICS. RIEMANN INVARIANTS /16

We shall start with equations (1.20):

$$\left.\begin{aligned} \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} &= -\nu\frac{\rho u}{r}, \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} &= 0, \\ \frac{\partial s}{\partial t} + u\frac{\partial s}{\partial r} &= 0. \end{aligned}\right\} \qquad (2.1)$$

Equation (2.1) contains different combinations of partial derivatives of ρ, u, p, s. Let us transform these equations, so that each equation contains derivatives of each function in the form of the following type of combination

$$\frac{\partial}{\partial t} + q\frac{\partial}{\partial r},$$

i.e., in the form of derivatives with respect to a certain direction dr = q dt, which is its own for each equation. In order that such a transformation be possible, system (2.1) must be a hyperbolic system, as is known. These directions do not depend on the method of reducing our system to this special form, and are called characteristic directions. The curve on the r, t plane, whose direction at each point coincides with the characteristic direction, is called the characteristic of the system (2.1). We thus obtain three systems of characteristics - with respect to the number of equations (2.1). In addition, after each of the differential equations is reduced to the form given above, it provides us with a specific differential relationship acting along the corresponding characteristic.

We should first point out the following. As is known, there are only two independent quantities among the thermodynamic quantities ρ, p, s, etc: by selecting any two thermodynamic quantities, we can generally speaking express any third one as a function of the first two. Each such relationship between three thermodynamic quantities is called an equation of state. It is a thermodynamic characteristic of the substance under consideration.

In particular, the pressure p is a function of the density ρ and the entropy s: /17

$$p = p(\rho, s).$$

Let us write the total differential of this function:

$$dp = p_\rho d\rho + p_s ds.$$

The partial derivative p_ρ is positive for all real substances, and represents the square of the speed of sound c:

$$p_\rho = c^2.$$

We thus have

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} = c^2\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r}\right) + p_s\left(\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial r}\right),$$

or, taking the third equation (2.1) into account, we have

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} = c^2\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r}\right)$$

Let us substitute $\frac{1}{c^2}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r}\right)$ in the equation of discontinuity, instead of $\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r}$, and let us then multiply this equation by $\frac{c}{\rho}$. We obtain

$$\frac{1}{\rho c}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r}\right) + c\frac{\partial u}{\partial r} = -\nu\frac{uc}{r}.$$

The Euler equation can be rewritten in the following form

$$\frac{1}{\rho c}c\frac{\partial p}{\partial r} + \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}\right) = 0.$$

Combining and subtracting the last two relationships, we obtain

$$\left.\begin{aligned}\frac{1}{\rho c}\left[\frac{\partial p}{\partial t} + (u+c)\frac{\partial p}{\partial r}\right] + \left[\frac{\partial u}{\partial t} + (u+c)\frac{\partial u}{\partial r}\right] &= -\nu\frac{uc}{r},\\ \frac{1}{\rho c}\left[\frac{\partial p}{\partial t} + (u-c)\frac{\partial p}{\partial r}\right] - \left[\frac{\partial u}{\partial t} + (u-c)\frac{\partial u}{\partial r}\right] &= -\nu\frac{uc}{r}.\end{aligned}\right\} \qquad (2.2)$$

The derivatives with respect to the direction dr = (u ± c)dt are within the brackets; therefore, equation (2.2) can be handled as follows. The following relationship holds with respect to the direction dr = (u + c)dt

$$\frac{dp}{\rho c} + du = -\nu\frac{uc}{r}dt,$$

and with respect to the direction dr = (u - c)dt - the relationship

$$\frac{dp}{\rho c} - du = -\nu\frac{uc}{r}dt.$$

The third equation of system (2.1) confirms the fact that, as we already know, /18 we will have the following with respect to the direction dr = u dt

$$ds = 0.$$

Thus, the lines on the plane r, t are defined by the equations dr = (u ± ± c)dt and dr = u dt, which represent characteristics of system (2.1). Let us write their equations one more time:

$$\left.\begin{aligned} dr &= (u+c)\,dt, \quad & \frac{dp}{\rho c} + du &= -\nu\frac{uc}{r}dt; \\ dr &= (u-c)\,dt, \quad & \frac{dp}{\rho c} - du &= -\nu\frac{uc}{r}dt; \\ dr &= u\,dt, & ds &= 0. \end{aligned}\right\} \qquad (2.3)$$

The first two equations contain the differentials of p and u. It is much more convenient to deal with relationships which include the differentials of entropy s and any other quantity. It is possible to transform equation (2.3) to this form, if we introduce the so-called Riemann invariants.

Generally speaking, the expression $\frac{dp}{\rho c}$ is not a total differential, since the quantity ρc depends not only on pressure p, but also on entropy s, for example. However, by adding an expression such as M ds* to $\frac{dp}{\rho c}$, as a result we obtain the total differential of a certain new thermodynamic quantity Φ. This may be done as follows.

Let us determine the thermodynamic quantity Φ by means of the relationship

$$d\Phi = \frac{dp}{\rho c} + M\,ds, \qquad (2.4)$$

where M is a certain thermodynamic quantity. A solution of the following equation is sufficient for determining Φ:

$$\frac{\partial\Phi}{\partial p} = \frac{1}{\rho c},$$

assuming that p and s are independent variables. Thus, Φ is determined within an accuracy of an arbitrary function.

By substituting $\frac{\rho dp}{c}$ from (2.4) in the first two equations (2.3), we obtain

$$d(\Phi \pm u) = -\nu\frac{uc}{r}dt + M\,ds.$$

The quantities

$$A = \Phi + u,$$
$$B = \Phi - u$$

are called Riemann invariants. Equations (2.3) assume the following form /19

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* Naturally, M must not be confused with the mass investigated in section 1.

$$\left.\begin{aligned} dr &= (u+c)\,dt, \quad & dA &= -\nu\frac{uc}{r}\,dt + M\,ds; \\ dr &= (u-c)\,dt, \quad & dB &= -\nu\frac{uc}{r}\,dt + M\,ds; \\ dr &= u\,dt, & ds &= 0. \end{aligned}\right\} \tag{2.5}$$

One distinguishing feature of equations (2.5) lies in the fact that the relationship along the characteristic includes, on the one hand, differentials of A, B, s, which we may regard as the desired functions and, on the other hand, the differentials of the coordinates t, r (it can be readily seen that the differential ds can be expressed by the coordinate differential). This facilitates the change to Riemann invariants.

As was already indicated, the definition of Φ, as well as the Riemann invariants, is very arbitrary. Their specific selection is dictated by practical considerations - from this point on we shall verify the fact that Φ is selected so that the quantity c is expressed most simply by Φ and s.

If Φ is selected so that $\Phi = 0$ in the case of $p = 0$, this quantity will have a simple physical meaning. Let us imagine a certain volume of substance, in an equilibrium state, contained between solid walls, and surrounded by empty space. If these walls are suddenly removed at any time, the substance will begin to scatter into the void. As will be seen at a later point, its leading boundary will move at a velocity of $u = \Phi$.

3. BASES OF THE METHOD /20

Let us introduce the following notation:

$$\alpha = u + c,$$
$$\beta = u - c,$$
$$F = -\nu\frac{uc}{r}.$$

Then the equations of characteristics (2.5) can be written in the following form

$$\left.\begin{aligned} dr &= \alpha\, dt, \quad dA = F\, dt + M\, ds; \\ dr &= \beta\, dt, \quad dB = F\, dt + M\, ds; \\ dr &= u\, dt, \quad ds = 0. \end{aligned}\right\} \tag{3.1}$$

We shall call the characteristics of the first set α-characteristics, of the second set - β-characteristics, and we shall call the characteristics of the third set trajectories.

We shall introduce the Lagrangian coordinate R by means of the relationship (1.17)

$$R^{\nu}dR = \frac{\rho}{\rho_0} r^{\nu}(dr - u\, dt). \tag{3.2}$$

Here ρ_0 is an arbitrary function of R, generally speaking. In the majority of cases, it is most convenient to set the following at a certain initial moment of time $t = t_0$:

$$\rho_0 = \rho.$$

We can assume R = r at this moment, and the Lagrangian coordinate will have the generally accepted physical meaning of the spatial coordinate of a given particle, which it had at the initial moment $t = t_0$. We shall designate $\frac{\rho}{\rho_0}$ by δ.

The relationship $dr = (u \pm c)dt$ or $dr - u\, dt = \pm c\, dt$ holds along the characteristics. Substituting this in (3.2), we obtain

$$R^{\nu}dR = \pm c\delta r^{\nu}dt,$$

or /21

$$dR = \pm c\delta\left(\frac{r}{R}\right)^{\nu}dt.$$

Let us set

$$R' = c\delta\left(\frac{r}{R}\right)^{\nu}.$$

We then have

$$dR = \pm R' dt. \tag{3.3}$$

The sign «+» pertains to α-characteristics, and the sign «-» pertains to β-characteristics.

The entropy s is a function of R. We should first note that all of the subsequent conclusions are absolutely independent of the fact that s is entropy in the same sense as it is used in thermodynamics. We may use any function of it, instead of s. Therefore, we shall call s an "entropy quantity" below.

We shall assume that the function s(R) is given.

Let us assume that we know the solution of system (2.1) in a certain region of the plane r, t, i.e., the functions of ρ, u, p, s are given which satisfy equations (2.1) and the equation of state of the given substance. We shall assume that these functions are twice differentiable continuously.

Let us examine two points which are rather close on the plane r, t, and let us designate them by the numbers 1 and 2. The number 3 designates the point lying on the intersection of the α-characteristic passing through the point 1, and the β-characteristic leading away from point 2 (Figure 6). At these points all of the quantities (u, p, ρ, s, α, β, etc.) will be designated by the corresponding numbers.

Our problem now consists of searching for the approximate formulas connecting the quantities at the point 3 with their values at the points 1 and 2. If we know points 1 and 2 (i.e., all of the quantities at them), these formulas enable us to compute point 3. Let us employ a method which is similar to the well-known Euler method which may be used in numerical integration of customary differential equations.

The following relationship holds along the α-characteristic: /22

$$dr = \alpha\, dt.$$

Let us now replace the variable α by the constant α_1. We obtain the approximate equation

$$dr = \alpha_1 dt,$$

from which it follows that

$$r_3 - r_1 = \alpha_1 (t_3 - t_1). \tag{3.4}$$

The second approximate equation may be written in exactly the same way

$$r_3 - r_2 = \beta_2 (t_3 - t_2). \tag{3.5}$$

Let us employ the approximate formulas (3.4) and (3.5) to compute the

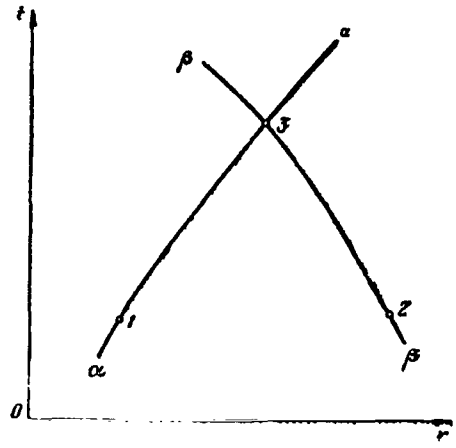

Figure 6

coordinates of point 3. Excluding the quantity r_3 from these formulas, we obtain

$$t_3 = \frac{\alpha_1 t_1 - \beta_2 t_2 + r_2 - r_1}{\alpha_1 - \beta_2}; \tag{3.6}$$

and by knowing t_3 we can compute r_3 by either of the formulas (3.4) (3.5).

Thus, the coordinates of point 3 have been computed (with a certain degree of accuracy). Let us now turn to equation (3.3). In absolutely the same manner as above, we can write the approximate equations

$$\left.\begin{aligned} R_3 &= R_1 + R_1'(t_3 - t_1), \\ R_3 &= R_2 - R_2'(t_3 - t_2). \end{aligned}\right\} \tag{3.7}$$

Any of them enables us to obtain the approximate value of R_3. Not having determined which of these equations we should use, we shall use any of them. Since by definition we know the functional dependence s = s(R), it is possible for us to compute s_3:

$$s_3 = s(R_3). \tag{3.8}$$

Let us now turn to equation (3.1). By replacing the differentials with the finite remainders in the relationships for the characteristics, just as previously, and the coefficients with the quantities from points 1 and 2 corresponding to them, we obtain two approximate formulas

$$\left.\begin{aligned} A_3 &= A_1 + F_1(t_3 - t_1) + M_1(s_3 - s_1), \\ B_3 &= B_2 + F_2(t_3 - t_2) + M_2(s_3 - s_2), \end{aligned}\right\} \tag{3.9}$$

from which we can calculate A_3 and B_3. /23

However, since $A = \Phi + u$, $B = \Phi - u$, we then have

$$\Phi = \frac{A+B}{2}, \quad u = \frac{A-B}{2}.$$

In particular,

$$\left.\begin{aligned} \Phi_3 &= \frac{A_3+B_3}{2}, \\ u_3 &= \frac{A_3-B_3}{2}. \end{aligned}\right\} \tag{3.10}$$

By knowing Φ_3 and s_3, we may compute any thermodynamic quantity. Thus, point 3 has been computed.

We may now significantly refine the result obtained, performing a second approximation or, as we shall call it, recalculation. We shall proceed as follows.

Instead of α_1, let us substitute the following quantity in formula (3.4)

$$\alpha_{13} = \frac{\alpha_1+\alpha_3}{2},$$

where α_3 is the quantity α at point 3 obtained as a result of the computation performed. Just as in formula (3.5), we substitute the following instead of β_2

$$\beta_{23} = \frac{\beta_2+\beta_3}{2}.$$

The new, approximate formulas which have been obtained along with formula (3.6), which has been changed, provide us with new, more accurate values for the coordinates of point 3.

Substituting the quantities R_1' and R_2' in formulas (3.7) by

$$R'_{13} = \frac{R'_1+R'_3}{2}; \quad R'_{23} = \frac{R'_2+R'_3}{2},$$

we can define the value of R_3 more precisely, and consequently that of s_3. Finally, a similar replacement of the coefficients in formulas (3.9) provides us with more accurate values of A_3 and B_3.

We shall show in the subsequent section that the formulas for the first approximation have residual terms on the order of h^2 (h - a step, i.e., the distance between points 1 and 2), and that the "recalculation" formulas have residual terms on the order of h^3. Thus, the recalculation increases the accuracy by one order of magnitude.

It should be possible to make a second recalculation by replacing the

quantities α_3, β_3, etc. in the coefficients of the approximate formulas by their values obtained as a result of the recalculation just performed. It can be shown, however, that such a second recalculation (as well as a third recalculation, and all subsequent recalculations) does not increase the order of magnitude of the residual terms. /24

We must now make certain definitions. Let us set the quantities α and β at a certain point on the r, t plane. The direction at this point is then called the timelike direction, if the double inequality holds along it

$$\beta < \frac{dr}{dt} < \alpha.$$

(since $c > 0$, then $\beta < \alpha$ always). If one of the inequalities is fulfilled in a given direction

$$\frac{dr}{dt} > \alpha, \quad \frac{dr}{dt} < \beta,$$

then the direction is called a spacelike direction (the direction $dt = 0$ belongs to the number of spacelike directions). If

$$\frac{dr}{dt} = \alpha \quad \text{or} \quad \frac{dr}{dt} = \beta,$$

then the direction is called the characteristic direction.

The nature of the given direction, whether it is timelike or spacelike, is determined by its location at one of the four sections of the plane, on which it is marked off by the characteristics passing through the given point. Thus, in Figure 7 the direction 1-2 is timelike, and direction 1-3 is spacelike.

The line on the r, t plane is called timelike or spacelike depending on /25 which direction it has at each point. The following graphic criterion may be employed to illustrate this. Let us draw the α and β characteristics "upward" from a certain point on the given line, i.e., toward an increase in t. If they are both located to one side of the line under consideration, then at this point it is spacelike (line AA, Figure 8). If these characteristics lie at different sides of our line (line BB), then this line is timelike.

Let us now set the line AB, at each point of which the quantities r, t, u, ρ, s are indicated (or any other total set of quantities). In addition, let this line be spacelike. Let us divide it by a series of points into very small sections. For each pair of adjacent points, we shall calculate a new point by means of the process described above. A new series is thus obtained, containing one point less. It can be readily seen that a line combining points in this new series will again be spacelike. By repeating this process, we obtain a new series and so on, until we arrive at a series consisting of one single point (Figure 9). We may determine the Cauchy problem solution for the system (2.1) within the curved triangle defined by the line for the initial data AB, by the α-characteristic emanating from point A, and by the β-characteristic emanating from point B. This triangle represents the region of influence for the AB line,

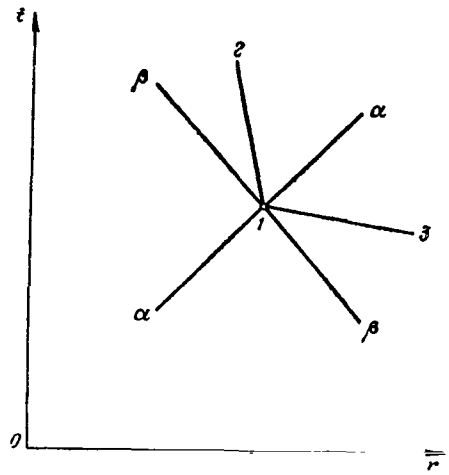

Figure 7

more correctly, it is the approximate expression of this region. It follows from the general theorems on hyperbolic equations that it covers the region in which the Cauchy problem may be solved with the initial data located on the AB section.

Up to this point, we have assumed that we are "producing" the characteristics "upward" by computing two points with a third, i.e., in the direction of an increase in t. From the physical point of view, this is the most reasonable. However, nothing is preventing us from proceeding in just the opposite manner, i.e., from "producing" the characteristic "downward". We may then formulate the solution in another triangle which has the same base AB and an apex which lies below this line. We must resort to this type of computation at times, although the computation is performed "from the bottom upward", as a rule. /26

We must frequently solve problems in which the line for the initial data includes the sections of the characteristics. As compared with the process described above, they do not present any new factors. Figure 10 presents examples of the grids of characteristics thus obtained. The problem in which the initial data are located on two characteristics (the left grid in Figure 10) is sometimes called the Goursat problem, and is even more typical than the classical Cauchy problem for the method presented, with the initial data on the spacelike line. The order of magnitude for the solution of this problem - and also, for example, a problem such as that whose grid is shown at the right in Figure 10 - is absolutely clear, and we shall not discuss it.

The case when the line of initial data is timelike does not have a physical meaning, as a rule. We shall not investigate it here.

Boundary conditions, the most important of which will be investigated below, are assigned on the timelike lines.

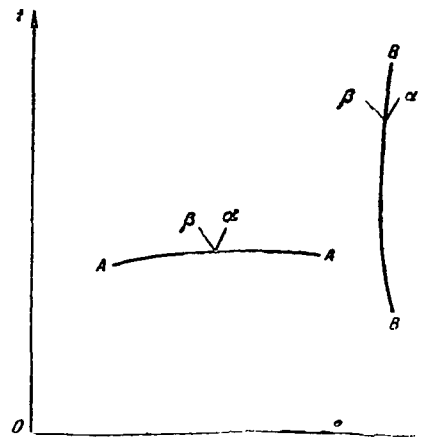

Figure 8

The method presented in this section is based on numerical integration of relationships (3.1), i.e., on integration of the Riemann invariants along the corresponding characteristics, in essence. However, this method is not obligatory - it would be possible to integrate equations (2.3) directly or to transform them into any other form. Actually, the method of integrating the Riemann invariants is not the most advantageous one in every case.

The analytic nature of the solution for equation (2.4) is entirely determined by the equation of state for the substance under consideration. Equations of state may be pointed out, for which Φ depends on c and s in such a complex manner that the introduction of Riemann invariants loses all practical meaning. It is true that all of those equations of state which we shall deal /27 with from this point on postulate the introduction of Riemann invariants (see section 5). However, the stipulation just presented must be always kept in mind when dealing with any other substance.

On the other hand, in every case when the equation of state permits it, it is recommended that the computation be performed in Riemann invariants. It is impossible to manage without them when computing expansion waves (section 8, 12). In the case of a flat, isentropic problem (ν = 0, s = const), equations (3.1) yield dA = 0 along the α-characteristics, and dB = 0 along the β-characteristics. The advantages of computing in Riemann invariants are indisputable here. If the entropy is not constant, but changes rather slowly, then the Riemann invariants will also change along the characteristics comparatively slowly, and their numerical integration will entail a comparatively small amount of error.

Based on these considerations, the rest of the discussion will be based on Riemann invariants, with very rare exceptions (see, for example, section 10).

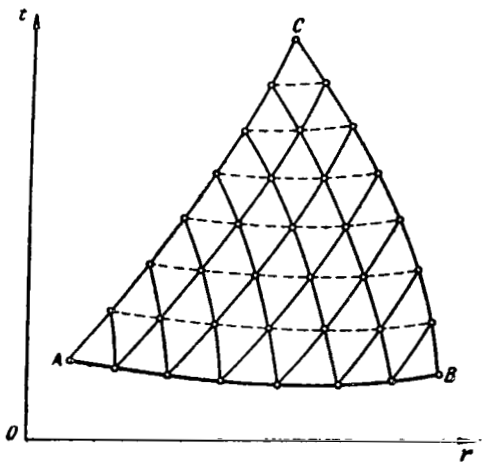

Figure 9

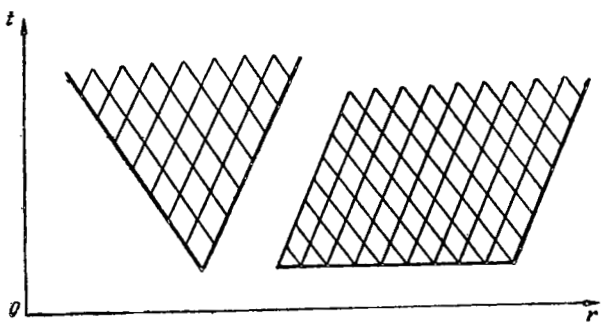

Figure 10

4. ORDER OF ACCURACY. SELECTION OF STEP. /28

We shall not derive the residual terms for the formulas of the preceding section; these terms are of no great practical importance. We shall confine ourselves to determining the order of accuracy of the formulas.

Just as previously, we shall assume that the quantities α, β, etc. in the region under consideration are functions which are differentiable (and, consequently, limited) twice continuously of the variables r, t. We shall assume that c is everywhere different from zero, and that the line connecting points 1 and 2 (see Figure 6) is a spacelike line. The differences $t_3 - t_1$ and $t_3 - t_2$ will be quantities on the order of h; the distances between points 1 and 2 have the same order of magnitude.

We have $dr = \alpha\, dt$ along the α-characteristic; consequently

$$r_3 - r_1 = \alpha_1 (t_3 - t_1) + \frac{\alpha'}{2}(t_3 - t_1)^2,$$

where α' is a derivative of $\frac{d\alpha}{dt}$ along the characteristic taken at a certain intermediate point. Consequently, we have

$$r_3 - r_1 = \alpha_1 (t_3 - t_1) + ah^2,$$

where a is a finite quantity. Similarly, we have

$$r_3 - r_2 = \beta_2 (t_3 - t_2) + bh^2.$$

Excluding r_3, we obtain

$$t_3 = \frac{\alpha_1 t_1 - \beta_2 t_2 + r_2 - r_1}{\alpha_1 - \beta_2} + \frac{b - a}{a_1 - \beta_2} h^2.$$

The denominator equals $(c_1 + c_2) + (u_1 - u_2)$ in the second term of the part on the right side, and in view of $c > 0$ it can be assumed to be different from zero. This term is entirely on the order of h^2. Returning to formula (3.6), we note that the first term in the part on the right side is an approximate value of t_3; let us designate it by t_3^*. In addition, we have

$$r_3 - r_1 = \alpha_1 (t_3 - t_3^* + t_3^* - t_1) + ah^2 = \alpha_1 (t_3^* - t_1) + \left(\alpha_1 \frac{b - a}{\alpha_1 - \beta_2} + a\right) h^2.$$

It directly follows that the approximate values of the coordinates for point 3, determined by formulas (3.4) - (3.6), differ from the real values by an amount on the order of h^2. A similar result is obtained for R_3. /29

We can determine s_3 on the basis of the functional dependence $s = s(R)$. In practice, the case when this dependence is defined analytically must be regarded as comparatively rare. If the function s(R) has no distinctive features, then the orders of accuracy of s and R will coincide.

Most frequently, however, this function is given in the form of a table, from which s_3 is obtained by interpolation. Let us assume that a step in this table is on the order of h; from this point on, we shall assume that this assumption is fully valid. The interpolation formulas must be selected so as not to decrease the order of accuracy of the quantities to be computed. It is known that a linear interpolation formula has a residual term on the order of h^2, and a quadratic term on the order of h^3. It thus follows that for the first approximation linear interpolation is sufficient for computing s_3. Quadratic interpolation is requisite in a recalculation.

In any case, s_3 is obtained in the first approximation with a residual term on the order of h^2. It can be readily seen that formulas (3.9) yield an error on the order of h^2 for A_3 and B_3.

In order to make a recalculation, it is necessary to compute α_3, β_3, R_3', F_3, M_3. They may be computed by means of s_3, A_3, B_3, t_3, r_3, i.e., they are functions of these five variables. If these functional dependences have no special features, then α_3, β_3, R_3', F_3, M_3 may be obtained with the same degree of accuracy.

Let us now investigate the recalculation process. On the basis of the quadratic formula of a trapezoid, we may write

$$r_3 - r_1 = \frac{\alpha_1 + \alpha_3}{2}(t_3 - t_1) - \frac{\tau''}{12}(t_3 - t_1)^2,$$

where α'' is the second derivative of $\frac{d^2\alpha}{dt^2}$ along the characteristic taken at a certain intermediate point. The quantity α_3 which is included in the first term on the right side is the exact value of α at point 3. We know its approximate value α_3^* which differs from the precise value by a quantity on the order of h^2:

$$\alpha_3 = \alpha_3^* + fh^2,$$

where f is a finite quantity. We thus have

$$r_3 - r_1 = \frac{\alpha_1 + \alpha_3^*}{2}(t_3 - t_1) + \frac{f}{2}h^2(t_3 - t_1) - \frac{\alpha''}{12}(t_3 - t_1)^3,$$

or, since $t_3 - t_1 \sim h$, /30

$$r_3 - r_1 = \frac{\alpha_1 + \alpha_3^*}{2}(t_3 - t_1) + mh^3.$$

Similarly, we have

$$r_3 - r_2 = \frac{\beta_2 + \beta_3^*}{2}(t_3 - t_2) + nh^3.$$

By repeating all of the previous procedures, we find that each quantity at point 3 will differ from its exact value by a residual term on the order of h^3, as the result of the recalculation. In particular, it follows that the recalculation

quantity - i.e., the difference between the second and first approximations - is on the order of h^2.

These estimates are purely local in nature. Thus, if we return again to Figure 9 and assume that the initial data on the line AB are defined absolutely precisely, we find that the dashed line closest to it will be computed with an error on the order of h^3. This means that if we divide the AB section into areas which are twice as small, the error of the quantities on the closest line decreases by a factor of approximately 8. This line will not coincide with the old line; it will be twice as close to the line AB.

If we are interested in the error at a certain fixed point in the plane r, t - for example, at the point C (Figure 9) - the situation will be somewhat different. If point C is located (with the given division) at a distance of m steps from the line AB, then -- in round numbers -- the error at this point will be m times greater than the error at the points closest to the line AB. With a decrease in the step h, the error at the closest points decreases proportionally to h^3, but the number of steps up to the point C increases proportionally to $\frac{1}{h}$, so that finally the error at point C decreases proportionally to h^2. Thus, the error at a fixed point on the plane is on the order of h^2.

This entire line of reasoning regarding the order of accuracy still does not provide us with the slightest idea of the actual error of the computational results. We shall begin the investigation of this problem by determining what advantage the complete expressions for the residual terms might have for us (if we knew them). In this connection, we would like to make a small digression.

By way of an example, let us examine any quadratic formula - for example, the formula of a trapezoid:

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(f_0 + f_1) - \frac{h^3}{12} f''.$$

the second term on the right is a residual term; the second derivative f" must be taken at a certain intermediate point which was previously unknown. How may we employ a residual term to estimate the error in the formula? /31

If the exact analytical expression for the function f(x) is known, such an estimate does not entail any particular difficulty. By differentiating the function of f twice and finding the maximum and minimum values of the second derivative of f" in the (x_0, x_1) segment, we may establish the upper and lower boundaries of the residual term. But how may this be, if we do not know the analytical form of the function of f (for example, if this function is given by the table)?

Several handbooks on numerical analysis recommend that the higher derivatives included in the residual term be estimated by means of suitable difference

formulas in this case. For example, in our case we may employ the approximate equation

$$f'' \approx \frac{f_0 - 2f_1 + f_2}{h^2}.$$

Using this formula to determine the second derivative, we may thus estimate the residual term.

If we substitute the approximate formula given above for the second derivative in the residual term of the trapezoid formula, we arrive at the following formula after several simplifications:

$$\int_{x_0}^{x_1} f\,dx = \frac{h}{12}(5f_0 + 8f_1 - f_2),$$

which in its turn represents a certain quadratic formula. Thus, the attempt to determine the residual term by means of a difference formula has led us simply to a calculation of the same integral by means of another, more precise, quadratic formula. The latter formula has its own residual term, and the entire question of determining the error simply eludes us even more, and has by no means been solved.

It does not follow from this, however, that such a method of determining the error has no value. On the other hand, a comparison of the results obtained by two different formulas presents a very strong argument in favor of the fact that the calculation was quite accurate. We need only note that this method cannot be regarded as completely faultless.

In addition, there is no point in employing only the formulas which are obtained by determining the derivatives included in the residual terms, in order to have such control. We may employ another formula with the same success, only if its order of accuracy is greater than the order of accuracy of the "working" formula. For example, in order to control the accuracy of the trapezoid formula, we may employ a quadratic formula such as the following: /32

$$\int_{x_0}^{x_1} f\,dx = \frac{h}{24}(-f_{-1} + 13f_0 + 13f_1 - f_2).$$

It is evident that we may apply such a method for determining the error to the process described in the preceding section. For this purpose, it is advantageous to employ a quadratic formula compiled on the basis of three non-equidistant points, which was pointed out by D. Yu. Panov*. We shall present it in a somewhat different form.

Let the values f_0, f_1, and f_2 of the function f(x) for the three values

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* Transactions of the V.A. Steklov Mathematical Institute (Trudy Matematicheskogo Instituta imeni V.A. Steklova). Vol. 38, 1951.

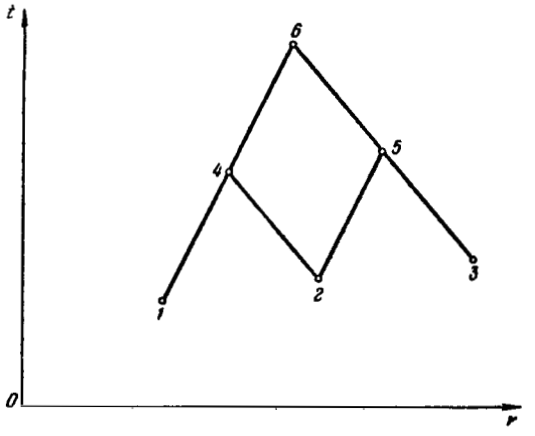


Figure 11

x_o, x_1, and x_2 of the independent variable x be designated by f_o, f_1, f_2. Let us calculate the following quantities:

$$f_0^* = \frac{f_2(x_0 - x_1) - f_1(x_0 - x_2)}{x_2 - x_1},$$

$$f_2^* = \frac{f_1(x_2 - x_0) - f_0(x_2 - x_1)}{x_1 - x_0}.$$

It can be readily seen that f_0^* is the result of linear extrapolation of the function f(x) to the value x_o using the points x_1, x_2. In exactly the same way, f_2^* is the result of the same extrapolation to x_2 using the points x_o, x_1. Let us now integrate the function f(x) over the (x_o, x_2) interval according to the trapezoid formula, taking f_o, f_2; f_o^*, f_2; f_o, f_2^* in turn as its value at the end of the interval:

$$I_1 = (x_2 - x_0)\frac{f_0 + f_2}{2},$$

$$I_2 = (x_2 - x_0)\frac{f_0^* + f_2}{2},$$

$$I_3 = (x_2 - x_0)\frac{f_0 + f_2^*}{2}.$$

The quadratic formula has the form

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{I_1 + I_2 + I_3}{3}. \qquad (4.1)$$

Let the points 4, 5, 6 (Figure 11) be now calculated from points 1,2, and 3. We can determine the error at point 6 by integrating the relationships along the α-characteristic with respect to points 1, 4, 6, and the relationships along the β-characteristic with respect to points 3, 5, 6. The divergence between the results obtained with the formulas in the preceding section and these control

calculations indicates (approximately) the magnitude of the residual terms in the first formulas.

There is no point in thus controlling each calculated point; this would /33 mean that we change from the formulas in the preceding section to different, more precise - but considerably more cumbersome - formulas. The basic calculation must be performed in the customary way, and the control method presented must be employed from time to time in order to trace the error.

One important stipulation must be formulated in connection with the use of quadratic formulas having increased accuracy. These formulas may only be effective if the functions to be integrated are quite "smooth", i.e., if α, β, R', F, M change quite evenly along the characteristic. If they undergo sharp jumps and bends (such cases are encountered quite frequently in practice), then the use of more complex quadratic formulas loses any meaning due to an excessive increase in their residual terms.

Another method of determining the error consists of repeating the calculation for a certain region by a different step. If it is found as a result that the step decrease does not lead to a significant change in the results, it may then be concluded that this step is sufficient within the framework of the accuracy assumed, i.e., the residual terms may actually be disregarded. If there is considerable divergence, then new, more precise values for the desired quantities may be obtained from a comparison of the results of these two miscalculations. Let a certain quantity f be calculated with a certain step h and with another step - for example, 2h. In the first case, the approximate value $f^{(1)}$ is obtained for it, and in the second case - $f^{(2)}$. Let the standard working formulas have residual terms on the order of h^2 (just as in our case). We can then write

$$f = f^{(1)} + mh^2,$$
$$f = f^{(2)} + 4mh^2,$$

where m is a certain coefficient. By excluding mh^2 from these two equations, we obtain

$$f = \frac{4f^{(1)} - f^{(2)}}{3}.$$

It is evident that this value will not be accurate (the values of m in the equations given above do not necessarily coincide), and that the residual term /34 for it will have a higher order of smallness.

Both methods presented for determining the error make it possible to select the requisite step at the beginning of the calculation, and then to control the accuracy from time to time. However, they are quite cumbersome. It would be very desirable to have more "operational" criteria.

One of these criteria is related to the possibility of computing the Lagrangian coordinate R both along the α-characteristic, and along the β-characteristic. The small difference between both results indicates to a certain extent that the correct step was chosen. Due to the fact that it requires a

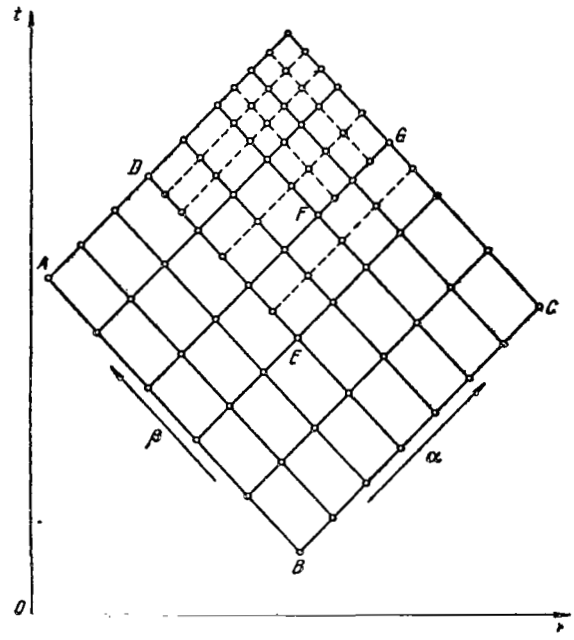


Figure 12

minimum of additional computations, this method is quite convenient, but its application is limited by another factor. The residual terms in formulas (3.7) contain higher derivatives of R' along the characteristics - for the first formula along the α-characteristic, and for the second formula, along the β-characteristic. If the derivatives in both directions are close to one another, the residual terms will be close, so that both formulas identically yield accurate results. On the other hand, if the higher derivatives of R' differ greatly along both sets of characteristics, then the residual terms will differ greatly in terms of magnitude. In this latter case, it is advantageous to compute R along the characteristic where R changes more "smoothly" - i.e., where its derivatives are smaller. To obtain complete agreement between both results derived from computing R, we must decrease the step above the power which is necessary to achieve the required accuracy (at least with respect to R). Thus, we must employ the criterion of the agreement between R calculated along unlike characteristics only when R' changes equally "smoothly" along both sets of characteristics.

The recalculation quantity-- i.e., the difference between two successive approximations of one and the same quantity - can provide the second criterion. The smallness of the recalculation represents a certain indication that the second approximation is close to the accurate value. As was shown above, the recalculation is on the order of h^2, while the residual terms are on the order of h^3. Therefore, there is no point in having the recalculation vanish in general (within the limits of the decimals retained) -- this would indicate a decrease in the step at which the necessity of recalculation is

generally eliminated. On the other hand, we have seen that the total computational error is on the order of h^2 - the same as the recalculation quantity. Therefore, in a certain sense it may be stated that the total error and the recalculation quantity are proportional to each other.

It is true that the coefficient of this proportionality depends most /35
strongly on the change in the quantities to be calculated, since the residual terms and the recalculation quantities are expressed in a completely different manner by means of the higher derivatives of the functions included in the computation. Therefore, it is absolutely impossible to formulate any universal criterion for a valid recalculation quantity. However, if a series of uniform problems is solved, a very careful analysis of the solutions for the former problems may establish the dependence between the recalculation quantity and the error in different regions, and this dependence may be utilized to solve the subsequent problems in the series.

In spite of the fact that each of the methods mentioned above for determining the error may not individually be called absolutely accurate or reliable, the combined use of them leads to almost reliable conclusions regarding the computational accuracy. We must now supplement this discussion with several practical considerations.

If a certain method indicates that the steps in a network of characteristics are too large, they must be decreased. Steps may be changed in both sets independently of each other. If, for example, recalculations of A increased excessively, then the step must be decreased along the α-characteristics, since A is integrated along them. The step is decreased by "inserting" additional points obtained by means of (quadratic) interpolation along the characteristic. We must thus take the fact into account that a change in the step along, for /36
example, the α-characteristic may be indicated in the recalculations of B, generally speaking.

By way of an example, let us examine Figure 12. Sections of the characteristics AB and BC serve as the initial data here. Let us assume that during the computational process it has been found that the steps on the DE section of the β-characteristic are large. If we employ the recalculation quantities in order to control the step, then - taking the fact into account that the recalculation quantity is proportional to the square of the step - we may compute a new step which is valid under the given conditions. Let us assume that the step must be decreased twofold at the two intervals which are closest to the point E, and that it must be decreased threefold on the third interval. By interpolation along the computed points, we may "insert" new points, from which new α-characteristics may come (they are shown by the dashed line in Figure 12). Such "insertion" is shown on the FG section of the α-characteristic.

The step must be increased as well as decreased during the computational process. If any factors indicate that a larger step is valid on the given section, it must be immediately increased, since an excessively small step leads, on the one hand, to an increase in the total amount of computational work and, on the other hand, produces the conditions for computational errors. The step is increased by simply discarding certain points.

5. COMPUTATIONAL FORMULAS FOR SPECIFIC EQUATIONS OF STATE

Let us commence with an ideal gas, i.e., a substance which satisfies the equation of state /37

$$p = k\rho^{\varkappa}. \tag{5.1}$$

Here $\varkappa$ is a constant (adiabatic index), and k is the entropy function k = k(s). Let us first find the Riemann invariants for this substance. We have

$$\frac{dp}{p} = \varkappa \frac{d\rho}{\rho} + \frac{dk}{k},$$

or

$$dp = \varkappa \frac{p}{\rho} d\rho + \frac{p}{k} dk.$$

We thus conclude that (see section 2)

$$c^2 = \varkappa \frac{p}{\rho}. \tag{5.2}$$

We find that equation (2.4) has the following solution

$$\Phi = \frac{2}{\varkappa - 1} c.$$

Actually, first of all we have

$$\frac{dp}{\rho c} = \frac{c}{\rho} d\rho + \frac{p}{k\rho c} dk. \tag{5.3}$$

In addition, it follows from (5.2) and (5.1) that

$$c^2 = \varkappa k \rho^{\varkappa - 1}$$

or, after differentiation,

$$2\frac{dc}{c} = (\varkappa - 1)\frac{d\rho}{\rho} + \frac{dk}{k},$$

and we thus have

$$\frac{c}{\rho} d\rho = \frac{2}{\varkappa - 1} dc - \frac{c}{(\varkappa - 1)k} dk.$$

Substituting this expression in (5.3,and allowing for the fact that $\frac{p}{\rho c} = \frac{c}{\varkappa}$, /38 we obtain

$$\frac{dp}{\rho c} = \frac{2}{\varkappa - 1} dc + \left(\frac{c}{\varkappa k} - \frac{c}{(\varkappa - 1)k}\right) dk = \frac{2}{\varkappa - 1} dc - \frac{c}{\varkappa(\varkappa - 1)k} dk,$$

or

$$d\left(\frac{2}{\varkappa-1}c\right)=\frac{dp}{\rho c}+\frac{c}{\varkappa(\varkappa-1)k}dk. \tag{5.4}$$

Making a comparison with (2.4), we find that we may assume

$$\Phi=\frac{2}{\varkappa-1}c. \tag{5.5}$$

Let us introduce the following notation

$$h=\frac{\varkappa+1}{\varkappa-1}.$$

We thus have

$$h+1=\frac{2\varkappa}{\varkappa-1},\quad h-1=\frac{2}{\varkappa-1},$$
$$\varkappa=\frac{h+1}{h-1}.$$

Consequently, we may write

$$\left.\begin{aligned}A&=(h-1)c+u,\\ B&=(h-1)c-u.\end{aligned}\right\} \tag{5.6}$$

In order to reduce the computational formulas in section 3 to a more convenient form, we may introduce a new entropy variable and a special function of pressure. In order to do this, we note that the following follows from (5.1) and (5.2):

$$\rho=\left(\frac{p}{k}\right)^{\frac{1}{\varkappa}},$$
$$c^2=\varkappa p\left(\frac{k}{p}\right)^{\frac{1}{\varkappa}}=\varkappa k^{\frac{1}{\varkappa}}p^{\frac{\varkappa-1}{\varkappa}},$$

i.e.,

$$c=k^{\frac{1}{2\varkappa}}\sqrt{\varkappa}p^{\frac{1}{h+1}}.$$

Consequently, the speed of sound c is represented in the form of the product of two coefficients; the first coefficient depends only on entropy, and the second coefficient depends only on pressure. Let us designate these coefficients as follows:

$$v=k^{\frac{1}{2\varkappa}},\quad z=\sqrt{\varkappa}p^{\frac{1}{h+1}}. \tag{5.7}$$

We then have

$$c=vz. \tag{5.8}$$

$$p = \varkappa^{-\frac{h+1}{2}} z^{h+1}. \tag{5.9}$$ /39

It can be readily seen that for ρ we obtain the formula

$$\rho = \varkappa^{-\frac{h-1}{2}} \cdot \frac{z^{h-1}}{v^2}. \tag{5.10}$$

If we introduce the notation

$$a^2 = \varkappa^{-\frac{h-1}{2}},$$

then formulas (5.1) and (5.10) can be written as follows:

$$p = \frac{a^2}{\varkappa} z^{h+1}, \quad \rho = a^2 \frac{z^{h-1}}{v^2}. \tag{5.11}$$

Differentiating the relationship $k = v^{2\varkappa}$ following from (5.7), we obtain

$$\frac{dk}{k} = 2\varkappa \frac{dv}{v}.$$

For the last term in (5.4), we thus obtain

$$\frac{c}{\varkappa(\varkappa-1)} \frac{dk}{k} = \frac{2\varkappa}{\varkappa(\varkappa-1)} \frac{c}{v} dv = (h-1) z\, dv.$$

Consequently,

$$d[(h-1)c] = \frac{dp}{\rho c} + (h-1) z\, dv,$$

and the relationships for the characteristics can be written in the following form

$$\left.\begin{aligned} dA &= F\,dt + (h-1) z\, dv, \\ dB &= F\,dt + (h-1) z\, dv. \end{aligned}\right\} \tag{5.12}$$

In addition, it can be readily shown that

$$\left.\begin{aligned} \alpha &= A - \frac{3-\varkappa}{4}(A+B), \\ \beta &= -\left[B - \frac{3-\varkappa}{4}(A+B)\right], \\ u &= \frac{A-B}{2}, \\ z &= \frac{A+B}{2(h-1)v}. \end{aligned}\right\} \tag{5.13}$$

Finally, for R' we obtain the following expression from (5.8) and (5.11):

$$R' = c\frac{\rho}{\rho_0}\left(\frac{r}{R}\right)^{\nu} = vz\frac{1}{\rho_0}a^2\frac{z^{h-1}}{v^2}\left(\frac{r}{R}\right)^{\nu} = \frac{a^2}{\rho_0}\frac{z^h}{v}\left(\frac{r}{R}\right)^{\nu}.$$

We may now write the complete system of computational formulas. Let us assume that we know the quantities t, r, R, A, B, u, v, z, α, β, F, R' at points 1 and 2. The same quantities at point 3 may be computed according to the formulas: /40

$$\left.\begin{aligned}
t_3 &= \frac{\alpha_1 t_1 - \beta_2 t_2 + r_2 - r_1}{\alpha_1 - \beta_2}, && (\alpha_1,\ \beta_2)\\
r_3 &= r_1 + \alpha_1(t_3 - t_1) = r_2 + \beta_2(t_3 - t_2), && (\alpha_1,\ \beta_2)\\
R_3 &= R_1 + R_1'(t_3 - t_1) = R_2 - R_2'(t_3 - t_2), && (R_1',\ R_2')\\
v_3 &= v(R_3)\\
A_3 &= A_1 + F_1(t_3 - t_1) + (h-1)z_1(v_3 - v_1), && (F_1,\ z_1)\\
B_3 &= B_2 + F_2(t_3 - t_2) + (h-1)z_2(v_3 - v_2), && (F_2,\ z_2)\\
\alpha_3 &= A_3 - \frac{3-\varkappa}{4}(A_3 + B_3),\\
\beta_3 &= -\left[B_3 - \frac{3-\varkappa}{4}(A_3 + B_3)\right],\\
u_3 &= \frac{A_3 - B_3}{2},\\
z_3 &= \frac{A_3 + B_3}{2(h-1)v_3},\\
F_3 &= -\nu\frac{u_3 v_3 z_3}{r_3},\\
R_3' &= \frac{a^2}{\rho_0}\frac{z_3^h}{v_3}\left(\frac{r_3}{R_3}\right)^{\nu}.
\end{aligned}\right\} \qquad (5.14)$$

The parentheses at the right contain the quantities which have been replaced by the average quantities in the corresponding formulas during the recalculation (for example, in the first formula we have used $\alpha_{13} = \frac{\alpha_1 + \alpha_3}{2}$ and $\beta_{23} = \frac{\beta_2 + \beta_3}{2}$ instead of α_1 and β_2). The notation $v_3 = v(R_3)$ indicates that the quantity v_3 is obtained from the previously known functional dependence $v = v(R)$ (for example, by interpolation according to the table).

The second equation of state which we shall examine differs very little from (5.1). It has the following form

$$p = a(k\rho^{\varkappa} - \rho_0^{\varkappa}), \qquad (5.15)$$

where $\varkappa$, α, ρ_0 are constants, and k is a function of entropy. In many cases it may be assumed that liquids and even (under certain conditions) solid bodies satisfy such an equation of state. One distinguishing feature of equation (5.15)

is the fact that the pressure p can vanish for non-zero k and ρ. This leads to certain phenomena (so-called separation) which are not characteristic of substances with the "gas" equation of state (5.1).

Equation (5.15) differs from (5.1) only due to the presence of a constant component. On the other hand, the pressure p is only included under the sign of the derivative in the differential equations (2.1). It thus follows that quantities which may be defined by relationships (5.6) may be used as the Riemann invariants for a substance with the equation of state (5.15). Just as always, the square of the speed of sound c is obtained by differentiating the pressure p over the density ρ (for a constant entropy variable k). This may be readily confirmed by performing the appropriate computations. /41

In order to formulate a complete system of computational formulas, for equation (5.15) we shall now determine the special variables which are, in a certain sense, similar to the variables z and v which we introduced for an ideal gas. In order to do this, let us first of all stipulate that ρ_0, which is included in the definition of the Langrangian coordinate (3.2), is equal to ρ_0 from equation (5.15). Then removing $\rho_0^{\varkappa}$ from the parenthesis in the right part of (5.15), we obtain

$$p = b(k\delta^{\varkappa} - 1), \tag{5.16}$$

where b is a new constant, and $\delta = \frac{\rho}{\rho_0}$.

According to equation (5.16), the pressure p vanishes in the case of $k = 1$ and $\delta = 1$. We shall call this state of the substance under consideration the initial state.

Differentiating (5.16) over $\rho = \rho_0\delta$, we obtain

$$c^2 = \frac{b}{\rho_0}\varkappa k\delta^{\varkappa-1}. \tag{5.17}$$

In particular, designating the speed of sound in the initial state by c_0, we have

$$c_0^2 = \varkappa\frac{b}{\rho_0},$$

or

$$b = \frac{\rho_0 c_0^2}{\varkappa}.$$

Therefore, equation (5.16) can be rewritten in the following form

$$p = \frac{\rho_0 c_0^2}{\varkappa}(k\delta^{\varkappa} - 1), \tag{5.18}$$

and relationship (5.17) can be rewritten in the form

$$c^2 = c_0^2 k\delta^{\varkappa-1} = c_0^2 (k\delta^{\varkappa})^{\frac{\varkappa-1}{\varkappa}} \cdot k^{\frac{1}{\varkappa}},$$

or

$$c = c_0 k^{\frac{1}{2\varkappa}} (k\delta^{\varkappa})^{\frac{\varkappa-1}{2\varkappa}}.$$

According to (5.18), the quantity $k\delta^{\varkappa}$ is a function of pressure p.

We can thus see that, just as previously, the speed of sound c is repre- /42
sented in the form of the product of two factors - one of which depends only on entropy, and the second of which depends only on pressure. We may set

$$v = c_0 k^{\frac{1}{2\varkappa}}, \quad z = (k\delta^{\varkappa})^{\frac{\varkappa-1}{2\varkappa}}.$$

If we define $h = \frac{\varkappa + 1}{\varkappa - 1}$, we then arrive at the following relationships

$$p = \frac{\rho_0 c_0^2}{\varkappa}(z^{h+1} - 1), \tag{5.19}$$

$$c = vz, \tag{5.20}$$

$$\delta = c_0^2 \frac{z^{h-1}}{v^2}, \quad \rho = \rho_0 c_0^2 \frac{z^{h-1}}{v^2}, \tag{5.21}$$

which are similar to formulas (5.8) and (5.11). The difference lies only in the constant component in the formula for p, and in the fact that we now have $\rho_0 c_0^2$ instead of the dimensionless quantity a^2. The reader can readily see that the computational formulas (5.14) remain in full force for the equation of state (5.15).

We should note that if v has a formal nature in the case of an ideal gas, it then has a specific physical meaning for a substance with the equation of state (5.15). Actually, it follows from (5.19) that condition p = 0 is equivalent to the condition z = 1; formula (5.20) shows that then v = c. Consequently, if we examine a certain volume of our substance which is in a balanced thermodynamic state, and then use a reversible adiabatic process (i.e., for constant v) to convert it to a state of zero pressure (or if we discharge it, as they say), then in this new state the speed of sound c equals v. We may therefore designate the latter quantity as the discharge speed of sound.

Let us investigate the following equation as an example of a more complex equation of state

$$p = b(\rho^{\mu} + k\rho^{\varkappa}). \tag{5.22}$$

Here b, μ, $\varkappa$ are constants and k is the function of entropy. This equation may be regarded as a type of "interpolation" equation - for small k the second term in the parenthesis may be disregarded, and the properties of the substance would be similar to the properties of an ideal gas with the adiabatic index μ.

On the other hand, for large k the second term plays the main role, and in terms of properties, our substance approximates an ideal gas with an adiabatic index $\varkappa$. We shall assume, for purposes of definition, that $\mu>\varkappa>1$.

Differentiating equation (5.22) with respect to ρ, we obtain /43

$$c^2 = b\mu\left(\rho^{\mu-1} + \frac{\varkappa}{\mu} k\rho^{\varkappa-1}\right). \tag{5.23}$$

Let ρ_0 be a certain constant initial density. As is customary, we designate the ratio $\frac{\rho}{\rho_0}$ by δ. In addition, let us introduce the following notation:

$$\mu - \varkappa = m, \quad \frac{\varkappa}{\mu}\rho_0^{-m}k = v, \quad \mu b\rho_0^{\mu} = a^2.$$

Formulas (5.22) and (5.23) then assume the following form

$$p = \frac{a^2}{\mu}\delta^{\mu}\left(1 + \frac{\mu}{\varkappa} v\delta^{-m}\right), \tag{5.24}$$

$$c^2 = \frac{a^2}{\rho_0}\delta^{\mu-1}(1 + v\delta^{-m}). \tag{5.25}$$

Let us first find the Riemann invariants. For this purpose, let us set

$$v\delta^{-m} = y^{-m},$$

so that

$$\delta = v^{\frac{1}{m}}y, \quad \rho = \rho_0 v^{\frac{1}{m}}y.$$

Substituting in (5.24) and (5.25), we may express p and c^2 by v and y:

$$p = \frac{a^2}{\mu} v^{\frac{\mu}{m}} y^{\mu}\left(1 + \frac{\mu}{\varkappa} y^{-m}\right), \tag{5.26}$$

$$c^2 = \frac{a^2}{\rho_0} v^{\frac{\mu-1}{m}} y^{\mu-1}(1 + y^{-m}). \tag{5.27}$$

Let us find the total differential of expression (5.26):

$$dp = a^2 v^{\frac{\mu}{m}} y^{\mu-1}(1 + y^{-m})\,dy + \frac{a^2}{m} v^{\frac{\varkappa}{m}} y^{\mu}\left(1 + \frac{\mu}{\varkappa} y^{-m}\right) dv.$$

In addition, by means of (5.27) we readily find that

$$\rho c = \sqrt{a^2\rho_0}\, v^{\frac{\mu+1}{2m}}\, y^{\frac{\mu+1}{2}}\sqrt{1 + y^{-m}}.$$

And we now have

$$\frac{dp}{\rho c}=\sqrt{\frac{\overline{a^2}}{\rho_0}}v^{\frac{\mu-1}{2m}}y^{\frac{\mu-3}{2}}\sqrt{1+y^{-m}}\,dy+\frac{1}{m}\sqrt{\frac{\overline{a^2}}{\rho_0}}v^{\frac{2\varkappa-\mu-1}{2m}}y^{\frac{\mu-1}{2}}\frac{1+\frac{\mu}{\varkappa}y^{-m}}{\sqrt{1+y^{-m}}}\,dv. \qquad (5.28)$$

We must separate the total differential from the right part of equation (5.28). In order to do this, we may determine the function $\phi(y)$ by means of the relationship /44

$$d\varphi=y^{\frac{\mu-3}{2}}\sqrt{1+y^{-m}}\,dy. \qquad (5.29)$$

The first term in the right part of (5.28) can then be rewritten in the following form

$$\sqrt{\frac{\overline{a^2}}{\rho_0}}v^{\frac{\mu-1}{2m}}\,d\varphi=d\left(\sqrt{\frac{\overline{a^2}}{\rho_0}}v^{\frac{\mu-1}{2m}}\varphi\right)-\frac{\mu-1}{2m}\sqrt{\frac{\overline{a^2}}{\rho_0}}v^{\frac{2\varkappa-\mu-1}{2m}}\varphi\,dv,$$

and, consequently,

$$\frac{dp}{\rho c}=d\left(\sqrt{\frac{\overline{a^2}}{\rho_0}}v^{\frac{\mu-1}{2m}}\varphi\right)+\frac{1}{m}\sqrt{\frac{\overline{a^2}}{\rho_0}}v^{\frac{2\varkappa-\mu-1}{2m}}\left(y^{\frac{\mu-1}{2}}\frac{1+\frac{\mu}{\varkappa}y^{-m}}{\sqrt{1+y^{-m}}}-\frac{\mu-1}{2}\varphi\right)dv.$$

Comparing the latter relationship with (2.4), we find that our goal has been achieved - $\frac{dp}{\rho c}$ is represented in the form of (2.4). We thus have

$$\Phi=\sqrt{\frac{\overline{a^2}}{\rho_0}}v^{\frac{\mu-1}{2m}}\varphi, \qquad (5.30)$$

$$M=\frac{1}{m}\sqrt{\frac{\overline{a^2}}{\rho_0}}v^{\frac{2\varkappa-\mu-1}{2m}}\left(y^{\frac{\mu-1}{2}}\frac{1+\frac{\mu}{\varkappa}y^{-m}}{\sqrt{1+y^{-m}}}-\frac{\mu-1}{2}\varphi\right). \qquad (5.31)$$

We may now perform numerical integration by the method presented in Section 3. In order to do this, we only need to have the table of the function $\phi(y)$ and, secondly, we must be able to compute M very simply with respect to the quantities v and Φ (in order to find α and β) and δ (in order to find R').

The function ϕ can be set equal to the following relationship, according to (5.29):

$$\varphi(y)=\int_0^y \eta^{\frac{\mu-3}{2}}\sqrt{1+\eta^{-m}}\,d\eta=\int_0^y \eta^{\frac{\varkappa-3}{2}}\sqrt{1+\eta^{m}}\,d\eta. \tag{5.32}$$

in the case of $\varkappa>1$, we will have $\frac{\varkappa-3}{2}>-1$, so that this integral converges. If it belongs to a number of Chebyshev functions which are expressed by means of elementary functions, we must turn to numerical integration in order to compute it. At the point y = 0, the function $\phi(y)$ may have a special characteristic. However, in the vicinity of this point it may be readily expanded in fractional powers of y.

For the computation of M, c and e, it is possible to employ the following relationships which follow from the definition of y and formulas (5.27), (5.30), (5.31): /45

$$\left.\begin{aligned}
&\sqrt{\frac{\rho_0}{a^2}}v^{\frac{1-\mu}{2m}}\Phi=\varphi(y),\\
&\frac{v}{\Phi}M=\frac{1}{m}\left[\frac{y^{\frac{\mu-1}{2}}\left(1+\frac{\mu}{\varkappa}y^{-m}\right)}{\varphi(y)\sqrt{1+y^{-m}}}-\frac{\mu-1}{2}\right],\\
&\frac{c}{\Phi}=\frac{y^{\frac{\mu-1}{2}}\sqrt{1+y^{-m}}}{\varphi(y)},\\
&\frac{c\delta}{\frac{1}{v^m}\Phi}=\frac{y^{\frac{\mu+1}{2}}\sqrt{1+y^{-m}}}{\varphi(y)}.
\end{aligned}\right\} \tag{5.33}$$

The right sides of relationships (5.33) are known functions of y; consequently, the left sides are interrelated by a functional dependence. We may compile a table in which $\sqrt{\frac{\rho_0}{a^2}}v^{\frac{1-\mu}{2m}}\Phi$, will be the argument, and the combinations $\frac{v}{\Phi}M$, $\frac{c}{\Phi}$, $\frac{c\delta}{\frac{1}{v^m}\Phi}$ will be the functions.

If v and Φ are known, we may then find $\frac{v}{\Phi}M$, and consequently M, from this

table. On the same line we then obtain $\frac{c}{\Phi}$, i.e., c. Finally, we find $\frac{c\delta}{v^{\frac{1}{m}}\Phi}$, which makes it possible to obtain the product ce which is requisite for computing R'.

With such a table at our disposal, no difficulties are entailed in the computational process. If the powers $\frac{1-\mu}{2m}$ and $\frac{1}{m}$ are fractional, it is also recommended that a table be complied in order to compute $v^{\frac{1-\mu}{2m}}$ and $v^{\frac{1}{m}}$.

When discussing the methods for computing shock waves, contact discontinuities and expansion waves at a later point, as a rule we shall confine ourselves to only ideal gas and avoid cumbersome formulas. However, the interested reader will have no difficulty in transferring these methods to other equations of state, employing the formulas presented here for computing the Riemann invariants.

6. SOME PRACTICAL CONSIDERATIONS /46

We are only investigating computations "by hand" which are primarily performed by either the customary calculating machine or one with an electric drive. Let us commence with the <u>form of the notation</u> of the intermediate results.

All of the notation is written on sheets of paper (preferably graph paper) which are laid out in rectangular squares. These squares are of such a size that the twelve numbers t, r, R, u, z, v, A, B, α, β, F, R' pertaining to one point [we are investigating the problem with the "gas" equation of state (5.1)] may be arranged inside each of them. For example, these numbers within the square may be arranged in the following order:

t	r	v	z
R'	R		u
		B	A
	F	β	α

The horizonal series of squares contains points lying on one α characteristic; the vertical series contains points lying on one β-characteristic. The coordinate t increases from left to right and from top to bottom at the points. The diagonals going from the left upward to the right represent spacelike lines; the quantities r and R increase in this direction. In particular, squares containing points 1, 2, and 3 (see Figure 6) will be arranged as follows:

	2
1	3

If the squares for points 1 and 2 are filled, then the computational formulas (5.14) make it possible to fill the square of point 3 successively. Let us present an example of this computation. Let us investigate the cylin- /47
drically symmetrical motion of an ideal gas with the adiabatic index $\varkappa = \frac{7}{5}$; we shall assume that e_0 equals unity. The dependence between the entropy quantity v and the Lagrangian coordinate R can be given by the formula

$$v = \left(\frac{5}{49}\right)^{\frac{5}{14}} R^{\frac{5}{7}}$$

(The reasons for selecting the function v(R) will be clarified below). Let us assume the following values for t, r, R, A, B at points 1 and 2:

$$t_1 = 1{,}0000, \quad t_2 = 1{,}0000,$$
$$r_1 = 1{,}0000, \quad r_2 = 1{,}0500,$$
$$R_1 = 1{,}00000, \quad R_2 = 1{,}05000,$$
$$A_1 = 2{,}6041, \quad A_2 = 2{,}7343,$$
$$B_1 = 1{,}1755, \quad B_2 = 1{,}2343.$$

It is primarily necessary to compute the quantities u, v, z, α, β, F, R' which are lacking for points 1 and 2. This may be done by the same formulas (5.14) and the well-known function v(R). We may then subsequently compute point 3. The results may be graphically presented as follows:

				1,0500	1,0000	0,45827	0,8660
				0,3969	1,05000		0,7500
						1,2343	2,7343
					—0,2835	0,3531	1,1469
				1	67	35	42
1,0000	1,0000	0,44258	0,8540	1,0739	1,0676	0,45063	0,8438
				29	467		86
0,3779	1,00000		0,7143	0,3617	1,02555		0,7190
						5	196
		1,1755	2,6041			1,1821	2,6202
					6	3	88
	—0,2700	0,3363	1,0923		—0,2546	0,3388	1,0993

Two values are given for each quantity at point 3 - the first approximation and the recalculation. In order to avoid making the notation more cumbersome, only the last decimals - which change during the recalculation - are written in the "recalculation" values. Thus, in the first approximation A_3 is equal to 2.6202, and after the recalculation it equals 2.6196. The quantity F_3 does not change as a result of the recalculation.

A specific amount of decimals are selected for each quantity, and the remaining ones are rounded off. When solving the problem of selecting the number of places, we must start with the requirement of over-all accuracy in the desired solution. This may be formulated in different ways. It is most reasonable /48 to impose the conditions of accuracy on the "physical" quantities u, p, ρ, but there are other possible approaches to this problem.

We feel it is most advantageous to have the requirement for accuracy imposed on quantities which directly participate in the computation - for example, A and B. The discussion presented below illustrates the solution for the problem of selecting the number of places under these conditions in the

numerical example presented above.

We shall start with the fact that the quantities A and B contain four decimals (i.e., four places after the comma). We should note first of all that terms such as F Δt are included in the formula for computing A and B. The quantity F is close to -0.3. This means that in order to provide for (with a certain margin) four correct decimals for the product F Δt, it is necessary to retain four decimals for t.

The quantity F is encountered only in the product F Δt. The differences Δt along both characteristics equal 0.0667, i.e., they have three significant digits. Consequently, it is sufficient to take four significant digits for F, because a further increase in the number of digits cannot increase the accuracy of the product F Δt.

In a similar manner, when investigating terms such as $(h-1)z\Delta v$, included in the same formulas, we find that we must take five decimals for v, and five significant digits for z.

The product α Δt has four accurate decimals (under the condition that α has no less than four significant digits). It thus follows that we must take four decimals for r. We thus obtain five decimals (the last is problematical) for R.

The quantities v and R are related by a functional dependence. When there is a change from point 1 to point 2, R changes by 0.05, and v changes by 0.01569. In round numbers we find that the increase in R is three times greater than the increase in v. This means that the fifth decimal in R may be three times less accurate than the fifth decimal in v. Consequently, in our case the requisite accuracy for v has been insured.

Since the increases in Δt have three significant digits, there is no point in selecting more than four significant digits for α, β, R'. On the other hand, as the reader may readily ascertain, the computational formulas absolutely insure this accuracy for these quantities under the conditions that A and B have four decimals. This pertains to the quantities F and z.

Thus, the problem of selecting the number of decimals has been solved. We should emphasize that the discussion has covered purely computational errors occurring due to the intermediate results being rounded off, and no mention has been made of the error entailed when the residual terms of the computational formulas (5.14) are discarded. This latter form of error was investigated /49 in Section 4; in our specific case, it may be discussed as follows.

We have taken points 1 and 2 from an exact solution of the system of equations (2.1) given by the formulas

$$\left.\begin{aligned} u &= \frac{5}{7}\frac{r}{t}, \\ \rho &= t^{-\frac{10}{7}}, \\ p &= \frac{5}{49} r^2 t^{-\frac{24}{7}}. \end{aligned}\right\} \tag{6.1}$$

The reader may readily see that these functions actually satisfy equations (2.1). He can readily verify the validity of the following relationships also:

$$\left.\begin{aligned} R &= rt^{-\frac{5}{7}}, \\ v &= \left(\frac{5}{49}\right)^{\frac{5}{14}} R^{\frac{5}{7}}, \\ A &= \frac{5}{7}(\sqrt{7}+1)\frac{r}{t}, \\ B &= \frac{5}{7}(\sqrt{7}-1)\frac{r}{t}. \end{aligned}\right\} \tag{6.2}$$

The equations for the characteristics have the following form:

$$\left.\begin{aligned} \alpha: \quad & r = \lambda_1 t^{\frac{6+\sqrt{7}}{7}}, \\ \beta: \quad & r = \lambda_2 t^{\frac{6-\sqrt{7}}{7}}, \end{aligned}\right\} \tag{6.3}$$

where λ_1 and λ_2 are arbitrary constants.

For the α-characteristic passing through point 1 (r=1, t=1), we have λ_1=1. For the β-characteristic passing through point 2 (r=1.05, t=1), we obtain λ_2=1.05. Solving equations (6.3) together, we find the exact coordinates of point 3 and then, substituting them in formulas (6.2), we find the quantities A, B, R, v at this point. The computations yield:

$$\begin{aligned} t &= 1{,}06667, \\ r &= 1{,}07304, \\ R &= 1{,}024695, \\ v &= 0{,}450359, \\ A &= 2{,}61966, \\ B &= 1{,}18256. \end{aligned}$$

After comparing these numbers with the results derived from numeri- /50
cal integration, we find that there is good agreement. The divergence which occasionally occurs in the unit of the last digit lies within the framework of computational error.

It is interesting to repeat these computations by means of another step - for instance, a duplicate step. In this case, we obtain

				1,1000	1,0000	0,47376	0,8776
				0,4158	1,10000		0,7857
						1,2931	2,8645
					—0,2970	0,3699	1,2015
				479	45	781	50
1,0000	1,0000	0,44258	0,8540	1,1512	1,1384	0,45899	0,8331
				95	4852		26
0,3779	1,00000		0,7143	0,3436	1,05230		0,7248
						87	40
		1,1755	2,6041			1,1872	2,6368
						04	49
	—0,2700	0,3363	1,0923			0,3424	1,1072

The formulas for the exact solutions lead to the following results:

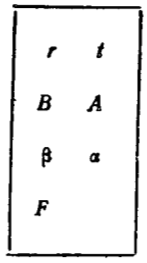
$$t = 1{,}13438,$$
$$r = 1{,}14765,$$
$$R = 1{,}048809,$$
$$v = 0{,}457903,$$
$$A = 2{,}63457,$$
$$B = 1{,}18929.$$

The divergence in the results is very distinct here, and thus the duplicate step is excessive.

The reader may readily ascertain that the recalculation quantities in the second example are exactly four times greater than in the first example, where they comprise six and four units of the last (fourth) decimal for A and B. We may thus conclude that if we had to solve the problem in which the change in the quantities is close to those given by formulas (6.1), then close to point r=1, t=1 (or one corresponding to it) recalculations of 5-10 units of the last decimal would result for A and B.

The computational formulas are significantly simplified in the case of so-called isentropic motion - i.e., motion when the entropy is the same throughout the entire mass of the substance. The function v(R) may be reduced to a constant, and the last terms vanish in the formulas for A and B. The necessity of computing v, R, R', z, u is thus eliminated.

The arrangement of the point thus assumes the following form /51



and the computational formulas are reduced to the following:

$$t_3 = \frac{\alpha_1 t_1 - \beta_2 t_2 + r_2 - r_1}{\alpha_1 - \beta_2}, \qquad (\alpha_1, \beta_2)$$

$$r_3 = r_1 + \alpha_1 (t_3 - t_1) = r_2 + \beta_2 (t_3 - t_2), \qquad (\alpha_1, \beta_2)$$

$$A_3 = A_1 + F_1 (t_3 - t_1), \qquad (F_1)$$

$$B_3 = B_2 + F_2 (t_3 - t_2), \qquad (F_2)$$

$$\alpha_3 = A_3 - \frac{3-\varkappa}{4}(A_3 + B_3),$$

$$\beta_3 = -\left[B_3 - \frac{3-\varkappa}{4}(A_3 + B_3)\right],$$

$$F_3 = \frac{\nu}{4(h-1)} \cdot \frac{B^2 - A^2}{r}.$$

In the flat case ($\gamma=0$) there is a further simplification, $F \equiv 0$, and the formulas for A and B change as follows:

$$A_3 = A_1,$$
$$B_3 = B_2.$$

It is true that Riemann has already studied this last case (flat isentropic motion) in detail. He indicated several cases when the solution may be expressed in closed analytical form*.

We may also note the case $\varkappa = 3$. As the computational formulas have shown, then $a=A$, $\beta=-B$, which also simplifies the calculation.

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* See Landau, L. V., Lifshits, Ye. M. Mechanics of Continuous Media (Mekhanika cploshnykh sred), Section 98, Moscow, 1953.

We already had occasion to note that problems in which the amount of points to be computed may amount to several thousand cannot be regarded as an exception. It is apparent that the problems of control, opportune discovery, and correction of errors is of very serious importance here. An error which is not discovered in time may entail the necessity of repeating the calculation for a large part of the problem, and sometimes for the entire problem as a whole.

The method of computing "on two hands" is extremely irrational, in /52 spite of a certain effectiveness, since it leads to duplication of the entire computational efforts. It is recommended only in individual, particularly important parts of the problem (for example, when computing the disintegration of the discontinuity). Methods which control the "smoothness" and "evenness" of the change in all the quantities along the characteristics are extremely advantageous for purposes of a "concurrent" and "working" control. It is clear that this method is only applicable when the initial data of the problem are quite "smooth."

If the steps along the characteristics are approximately the same, then the values for each of the quantities (A, B, v, u, z, et cetera) must change from point to point by approximately the same value. The changes in these increases must occur systematically. Each "jumping out" of any quantity points to the necessity of carefully checking the computation; if no errors are discovered, then the "jumping out" must be clarified.

It is recommended that the recalculation quantities be constantly watched during the process; in a certain sense, they not only indicate the residual terms, they also represent a very effective method for controlling the correctness of the calculations. As a rule, each computational error leads to an abnormal increase in the recalculations, and each "jumping out" of the recalculations must be immediately investigated. The recalculation quantities must not only be small, but they also must change "smoothly" from point to point, under the condition that the steps of the network also change "smoothly."

If an error, which was allowed at an earlier stage and remained unnoticed, is discovered during the calculational process, then the entire region between the characteristics emanating from this erroneous point (i.e., the propagation region of this point) must be calculated again. However, if the error is small, so that its square may be disregarded, then the correction of the results may be frequently simplified considerably.

From the general theory of hyperbolic systems, it is known that small perturbations are propagated along the characteristics, with no change in their magnitude (in the first approximation). The perturbation of A is thus propagated along the α-characteristic, and the perturbation of B is propagated along the β-characteristic; perturbation of R and v is propagated on the trajectory. Therefore, after the erroneous point is corrected, the increase in A must be added to the values of A at all points of the α-characteristic

emanating from the corrected point. The same procedure must be followed for B. The situation is somewhat more complex for R and v, since R is not integrated on the trajectory, but along the characteristics. If only v is incorrect, then this error will not be propagated in general, since at all other /53 points v is again determined from the functional dependence v(R). The quantity R must be corrected along those characteristics over which it was integrated.

It is recommended that all the points to be computed be plotted on a graph. A graph of r, t, which presents a particularly graphic illustration of the computational procedure, is one of the main graphs. The scales along both axes must be selected so that the distances between adjacent points are not small, in order to make an accurate determination of the direction of the characteristic section between them. These sections are drawn along a ruler, and the entire network of characteristics is laid out on the graph during the computational process.

It is also recommended that graphs of A, t and B, t be drawn in order to provide more comprehensive control. It is true that the features of these graphs are complex as compared with the graph of r, t (they may "be superimposed on each other"), but they make it quite easy to verify the "local" smoothness of the quantities.

Different factors may be encountered during the computational process. We shall indicate two of them.

If the characteristics of one and the same set intersect each other, this indicates the development of a shock wave (when there are no computational errors). The steps which must be taken in this case will be presented in the following sections.

As was already indicated previously, the pressure can drop to zero in a substance with an equation of state such as (5.15). In addition, it may become negative. As a rule, this indicates the development of a so-called separation; the methods for computing it will be presented at a later point.

7. BOUNDARY CONDITIONS /54

The problems of gas dynamics do not always hold for the entire infinite space filled with gas or liquid. Very frequently it is necessary to compute the motion of a substance filling a region which is limited in space. One typical case is the motion of a gas in a cylindrical tube which is covered at both sides with plungers. Moving, according to a certain law, these plungers transmit motion to the substance located between them. The law underlying the motion of the plunger cannot be given as the dependence of its coordinate on time. For example, the plunger may sustain a specific pressure at its surface. In particular, depending on the conditions of the problem, this pressure may be zero; we are then dealing with a substance scattering into space. In one way or another, a boundary condition arises on the plunger surface.

One characteristic feature of these boundary conditions is that the boundary moves according to the law $\frac{dr}{dt} = u$ on the r,t plane, i.e., it is the trajectory. In other words, there is no flow of the substance through the boundary. This factor presents certain advantages when computing the motion of a substance adjacent to the boundary; these advantages lie in the fact that the entropy of the particles adjacent to the boundary remain constant. Therefore, only one condition is sufficient for making a complete determination of the motion. However, it must be noted that this type of boundary condition is not /55 uniquely possible; we shall verify this at a later point. At this point we shall only investigate a boundary with satisfies the condition $\frac{dr}{dt} = u$ or, which is the same thing, R = const.

Let us assume that the boundary defining a substance (for purposes of definition) to the right is given (with its boundary condition). Its motion may be computed in the following manner.

Let us assume that point 1 on the boundary (Fig. 13) has already been calculated. We may then compute the entire β-characteristic 1 - 1' in the usual way. Let us place a certain point 2 on this characteristic. Let us draw a straight line $\frac{dr}{dt} = u_1$ from point 1, and the straight line $\frac{dr}{dt} = \alpha_2$ from point 2, and let us determine the point at which they intersect. This will be (in the first approximation) the following point on the boundary which we shall designate by the number 3. All the remaining terms in it may be calculated as follows. We first have $R_3 = R_1$ and $v_3 = v_1$. In addition, we select the following relationship along the α-characteristic 2 - 3:

$$A_3 = A_2 + F_2(t_3 - t_2) + (h - 1) z_2 (v_3 - v_2). \qquad (7.1)$$

Since we know t_2 and v_3, we may find A_3 directly. It is necessary to include the boundary condition in order to compute B_3.

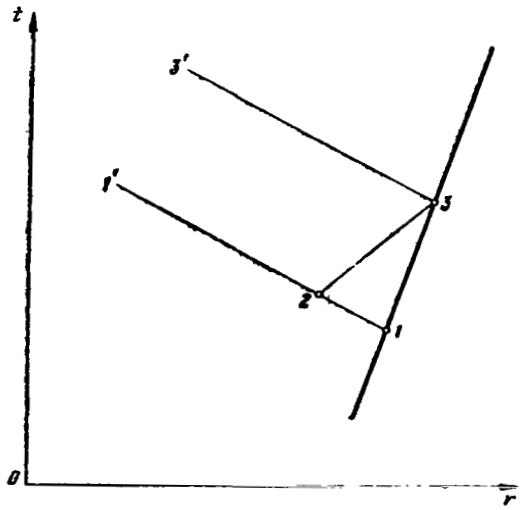


Figure 13

Let us formulate this condition as u = u(t) (on the boundary). Then, knowing t_3, we can immediately obtain u_3 and then B_3, because - as may be readily seen -

$$B_3 = A_3 - 2u_3. \tag{7.2}$$

After this, the "formal calculation" of the remaining terms (z, α, β, F, R') is performed in the usual way.

If the boundary condition is given in the form of the function p=p(t), we may immediately obtain the quantities z and c at point 3:

$$\left.\begin{aligned} z_3 &= \left(\frac{\varkappa}{a^2}\,p_3\right)^{\frac{1}{h+1}}, \\ c_3 &= \frac{z_3}{v_3}. \end{aligned}\right\} \tag{7.3}$$

We now find

$$B_3 = 2(h-1)c_3 - A_3. \tag{7.4}$$

The recalculation is performed in the usual way. The coordinates at point 3 are first defined more accurately on the basis of the equations

$$r_3 - r_1 = u_{13}(t_3 - t_1),$$
$$r_3 - r_2 = \alpha_{23}(t_3 - t_2)$$

then the quantities F_2 and z_2 may be replaced by the averaged quantities /56
(F_{23} and z_{23}) in formula (7.1), after which B_3 may be computed either from (7.2), or from (7.3) and (7.4).

If the boundary condition is not given in the form of the function u(t) or p(t), then formulas (7.2) - (7.4) naturally change, but the same goal remains - B must be obtained on the basis of the specific v and A by means of the boundary condition.

B is integrated for the "left" boundary along the characteristic; A is obtained from v, B and the boundary condition.

Let us give the total set of computational formulas for an ideal gas and a boundary condition having the form u(t) ("the right" boundary):

$$\left.\begin{aligned}
t_3 &= \frac{u_1 t_1 - \alpha_2 t_2 + r_2 - r_1}{u_1 - \alpha_2}, && (u_1,\ \alpha_2)\\
r_3 &= r_1 + u_1(t_3 - t_1) = r_2 + \alpha_2(t_3 - t_2), && (u_1,\ \alpha_2)\\
R_3 &= R_1,\\
v_3 &= v_1,\\
A_3 &= A_2 + F_2(t_3 - t_2) + (h-1)z_2(v_3 - v_2), && (F_2,\ z_2)\\
u_3 &= u(t_3),\\
B_3 &= A_3 - 2u_3,\\
\alpha_3 &= A_3 - \frac{3-\varkappa}{4}(A_3 + B_3),\\
z_3 &= \frac{A_3 + B_3}{2(h-1)v_3},\\
F_3 &= -\nu\frac{u_3 v_3 z_3}{r_3}.
\end{aligned}\right\} \qquad (7.5)$$

We must now perform the recalculation (the quantities to be averaged are given in the parentheses), after which we may additionally calculate

$$\left.\begin{aligned}
\beta_3 &= -\left[B_3 - \frac{3-\varkappa}{4}(A_3 + B_3)\right],\\
R_3' &= \frac{a^2}{\rho_0}\frac{z_3^h}{v_3}\left(\frac{r_3}{R_3}\right)^{\nu}.
\end{aligned}\right\} \qquad (7.6)$$

For purposes of control, it is advantageous to integrate R along the section of the α-characteristic 2-3:

$$R_3 = R_2 + \frac{R_2' + R_3'}{2}(t_3 - t_2). \qquad (7.7)$$

and to compare the result obtained with the quantity transferred from point 1.

Point 2 on the 1-1' characteristic is chosen from the general considerations related to the assumed magnitude of the step. Just as previously, /57 we may employ the recalculation quantity as an additional criterion, as well as the divergence R integrated according to formula (7.7) with the value of R transferred from point 1. As point 2, it is more advantageous to take one of the calculated points - for example, the one closest to the boundary. However, this is not always possible, since the step along the boundary may be extremely large. We must then resort to "insertions" close to the boundary.

The condition p=0 corresponding to expansion of the substance into space is an important special case of the boundary condition. If the equation of state always has the form (5.15), then this condition is equivalent to z=1. We shall call this boundary the free boundary.

Since z=1, then c=v. Thus, both u and c along the free bounary were known previously. Therefore, after integration of A (in the case of the "right" free boundary), B is obtained from the following formula

$$B = 2(h-1)v - A.$$

The remaining computation is performed in the usual way on the basis of formulas (7.5).

In the case of an ideal gas, condition p=0 leads (for finite v) to the equation c=0, and in addition to α=β. The boundary is a special line, since the α-characteristics touch the β-characteristic upon this line. It is true that frequently the motion of the boundary itself is determined very simply - if the derivative $\frac{dc}{dr}$ at the boundary is finite, then its velocity is constant. However, the region directly adjacent to it cannot be computed by the method of characteristics.

8. EXPANSION WAVES /58

Let us first investigate flat isentropic motion, i.e., the case of $\nu = 0$, $v = \text{const}$. The region on the r,t, plane in which one of the Riemann invariants (A or B) retains a constant value is called a simple wave.

For example, let us set $B = \text{const}$ in the simple wave. Let us investigate a certain (arbitrary) α-characteristic. Since the corresponding Riemann invariants are always constant along the characteristics in the flat isentropic case, along our α-characteristic $A = \text{const}$ will hold. Thus, since $B = \text{const}$, it immediately follows that any quantity (except for the coordinates t, r, R), particularly α, will be constant along this characteristic. And since the equation of the characteristic is $\frac{dr}{dt} = \alpha$ the characteristic will be a straight line. Similarly, there will be rectilinear β-characteristics in the simple wave with the condition $A = \text{const}$. We shall call these characteristics longitudinal (i.e., the α-characteristics in the case of $B = \text{const}$ and the β-characteristics in the case of $A = \text{const}$). We shall the characteristics of the opposite set transverse characteristics. We should point out that the transverse characteristics must by no means be rectilinear.

A simple wave is called a centered wave, if its longitudinal characteristics pass through one point (apex of the wave). Figure 14 shows two types of centered simple waves. After convergence, the first type of wave changes into a shock wave, but absolutely not into a simple wave of the second type.

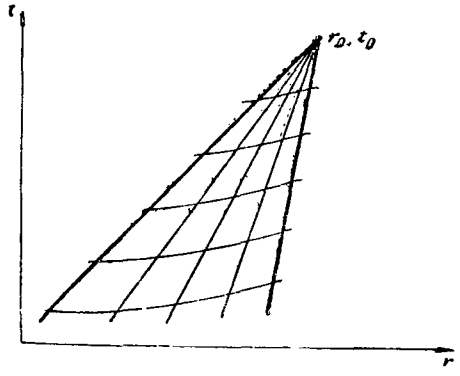


Figure 14

For purposes of definition, let us investigate a centered simple wave in which $B = \text{const}$. We shall use r_0, t_0 to designate the coordinates of the wave apex. Since α is the angular coefficient of the tangent to the α-characteristic, and these characteristics are rectilinear, at any point on our wave we have

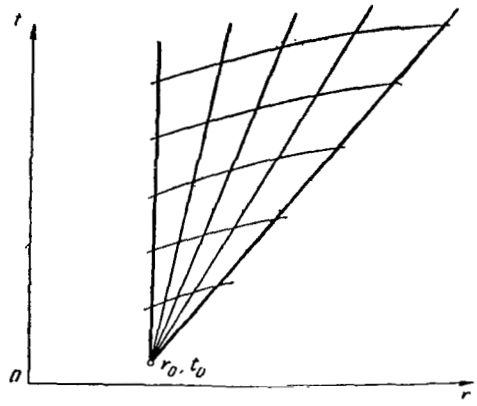


Figure 14a

$$\alpha = \frac{r - r_0}{t - t_0},$$

or, assuming that /59

$$\frac{r - r_0}{t - t_0} = \xi,$$

we obtain

$$\alpha = \xi. \tag{8.1}$$

Expressing α and β by c and u, we obtain

$$\begin{aligned} u + c &= \xi, \\ (h-1)c - u &= B, \end{aligned}$$

and solving for u and c, we obtain

$$\left.\begin{aligned} u &= \frac{h-1}{h}\xi - \frac{1}{h}B, \\ c &= \frac{1}{h}\xi + \frac{1}{h}B. \end{aligned}\right\} \tag{8.2}$$

Since h>0 always, c increases along with ξ. It can be readily seen that in the first type of wave (Figure 14) ξ also increases with an increase in t for constant r. In the second type of wave, the opposite is true, and ξ decreases

with an increase in t in the case of r = const. The quantity c changes at the same side, and since v = const, it can be readily determined on the basis of formulas (5.11) that in the first case p and ρ also increases with an increase in t (for constant r), and in the second case, the opposite is true and they decrease. Therefore, a simple wave of the first type is called a compression wave, and a simple wave of the second type is called an expansion wave.

Formulas (8.2) provide a complete solution of equations (2.1) in a centered, simple wave. Actually, we may express any quantity by u, c, and v (we should remember that v is the given constant). For example, it can be readily shown that /60

$$A = 2\frac{h-1}{h}\xi + \frac{h-2}{h}B,$$
$$\beta = \frac{h-2}{h}\xi - \frac{2}{h}B.$$

The latter relationship makes it possible for us to find the equation for the transverse characteristics. By integrating the equation

$$\frac{dr}{dt} = \beta,$$

after simple computations we obtain

$$r - r_0 = \lambda |t - t_0|^{\frac{h-2}{h}} - B(t - t_0),$$

where λ is an arbitrary constant.

From this point on, we shall only be interested in an expansion wave. Let us investigate the moment t_0 at which it is formed (Figure 15). Since the quantities u, c, ρ, p, et cetera, have different values, generally speaking, on the longitudinal characteristics defining it, at the moment $t = t_0$ these quantities will undergo a discontinuity at the point corresponding to the wave apex. A graph of the function u(r) will have the form shown in Figure 16 at this moment.

However, the quantities v and B, being constant over the entire region occupied by the expansion wave, will not undergo a discontinuity.

In addition, it follows from (8.2) that u is always greater to the right of the discontinuity than it is to the left (this is valid not only for the "B-wave" under consideration, but also for the "A-wave"). /61

Thus, in order that the expansion wave be formed from the discontinuous initial data, the following three conditions must be fulfilled:

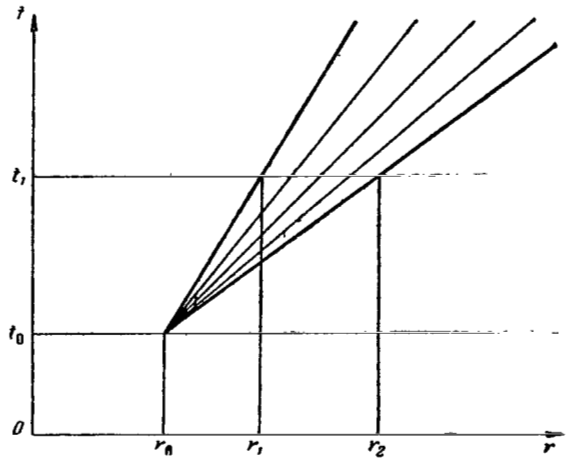


Figure 15

(a) There is no jump in the entropy at the discontinuity point;

(b) One of the Riemann invariants (A or B) is also continuous;

(c) The quantity u is greater to the right of the discontinuity than it is to the left.

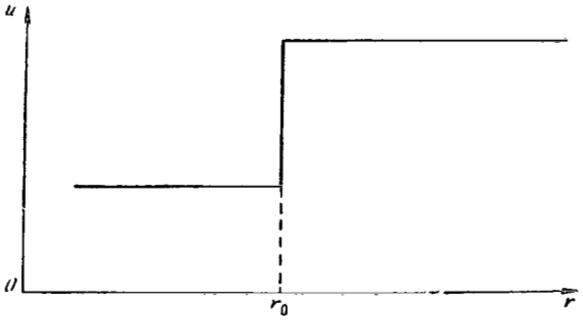


Figure 16

If one of these conditions were disturbed, the entire phenomenon would be more complex.

The discontinuity producing the expansion wave exists for only one instant; all of the quantities will be continuous at any subsequent moment, for example, at the moment $t = t_1$ (Figure 15). A graph of the velocity u will have,

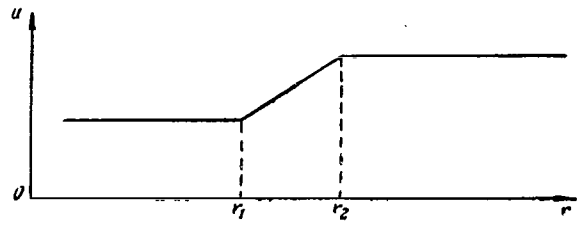


Figure 17

for example, the same form (Figure 17). The quantity u is continuous at the points r_1 and r_2, and its derivative $\frac{\partial u}{\partial r}$ undergoes a discontinuity. Discontinuities of this type are called first-order discontinuities. Thus, the outer longitudinal characteristics of an expansion wave represent first-order discontinuities.

We indicated above the three conditions under which an expansion wave may develop from the initial discontinuity. Are these conditions sufficient?

If there is a region near the expansion wave, within which the solu- /62 tion is given by formulas such as (8.2), then it must be stated that these conditions are insufficient. Additional restrictions must be imposed on the discontinuity - namely, along one of its sides the entropy and the corresponding Reimann invariants are constant.

However, if these latter conditions are not fulfilled, the motion after the discontinuity will have the same nature from the qualitative viewpoint. The longitudinal characteristics, not being rectilinear, will emanate from one point just as before (Figure 18). The entropy and the Reimann invariant, not being constant, will change very slightly close to the wave apex. The outer longitudinal characteristics will be lines of first-order discontinuities, et cetera.

The motion will have the same nature not only in the plane, but also in the cylindrical and spherical cases. It is only important that the initial discontinuity satisfy the three conditions indicated above.

Therefore, from this point on we shall use the term expansion wave to designate the region on the r, t- plane occupied by the characteristics emanating from the discontinuity point satisfying the three conditions formulated above.

Within such a "generalized" expansion wave, u and c will not satisfy the relationships (8.2). Nevertheless, these formulas provide the main terms of

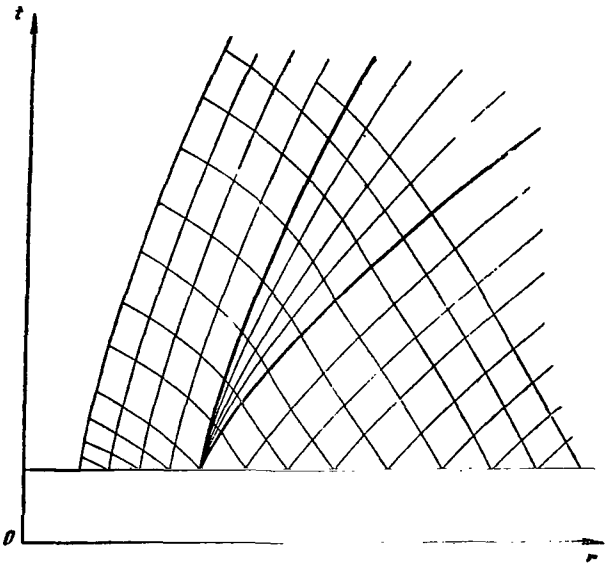


Figure 18

the functions u(r, t) and c(r, t). In the general case, the function /63
u(r, t) within the expansion wave is expanded in series such as the following:

$$u(r, t) = f_0(\xi) + (t - t_0) f_1(\xi) + (t - t_0)^2 f_2(\xi) + \ldots,$$

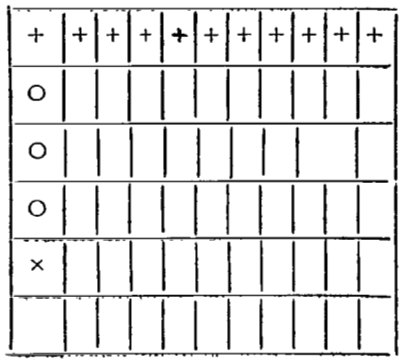
and the function $f_0(\xi)$ coincides with the right part of the first formula (8.2).

All of the remaining terms permit similar expansions.

The expansion wave apex is a particular point, since the quantities u, c, ρ, p, et cetera, along different characteristics have different limiting values at this point. Nevertheless, no particular difficulties are entailed in calculating the expansion wave.

For example, let us calculate the expansion wave shown in Figure 18. The region between the outer first-order discontinuities is filled with an infinite number of α-characteristics emanating from the wave apex. In performing this computation, we must confine ourselves to a certain finite number of characteristics. It is impossible to give a universal formula for selecting this number. Experience must be the guide in each separate case, and sometimes trial computations for a small region close to the apex must serve as a guide. Let us assume that we have decided to "release" five characteristics (including the outer characteristic).

The region located to the right of the expansion wave can be calculated in the customary way up to the outermost, right characteristic of the wave, inclusively. Let us take a piece of paper, divide it - as is customary - into squares, and let us draw the points for the outermost, right α-characteristic in the upper horizontal line (these points are designated by crosses in this diagram). In the far left vertical column, skipping three squares, let us draw the point corresponding to the left side of the initial discontinuity - i.e., the initial point of the outer, left wave characteristic (designated by X). Then, in the three intermediate squares (designated by circles) let us draw the points compiled in the following way:



The quantities t, r, R, v, B at these points coincide with and /64
equal the corresponding quantities at the points already drawn in the left vertical columns (at which they must coincide with each other, according to the conditions in the initial discontinuity). With respect to A, at these three points the values lying between the upper (+) and the lower (X) points are recorded, so that the values of A differ from each other by the same amount at each of the two subsequent points in the left vertical column.

The remaining terms (z, u, α, β, R', F) can be computed according to general rules.

We now have two series of points, formally comprising the initial data for the Goursat problem. Using general rules, we may compute all of the points located in the four horizontal lines beginning with the points which are designated by circles and X. The expansion wave will be computed in this way.

The question already arises at the beginning of the calculation as to whether the number of characteristics we have "released" from the wave apex is sufficient. If the recalculation (or other indications) indicates that the steps are excessively large, we must then repeat the initial division, increasing the amount of intermediate points. If, on the other hand, the steps are excessively small, it is recommended that the "discarding" of the points is not done in haste, since the steps initially increase very rapidly along the transverse characteristics in the expansion wave. It is recommended that the

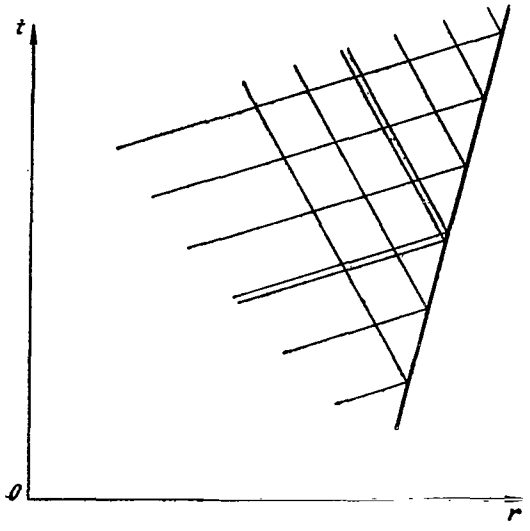


Figure 19

initial division be performed with a certain reserve, since, as a rule, /65 it is subsequently necessary to resort to "insertions" which are frequently repeated.

The outer characteristics of the expansion wave are first-order discontinuities. In practice, this means that interpolation cannot be performed through these lines; when selecting the points for interpolation, one must ascertain that the mean point does not lie on the first-order discontinuity line. Finally, interpolation is fully admissible along the first-order discontinuity.

In view of this fact, it must be particularly noted that the first-order discontinuities are not confused with the customary characteristics. For example, they may be circled with a red pencil on the r, t-graph. It is also recommended that the corresponding characteristics be designated on the pages being used.

If the first-order discontinuity reaches the boundary described in Section 7, it is "reflected" from it by the first-order discontinuity (see Figure 19, where the first-order discontinuity is designated by the double line).

We must also keep the fact in mind that with the passage of time the first-order discontinuities have a tendency to weaken, so that the discontinuities of the functions for them become less significant. Therefore, it is frequently possible to disregard the "old" first-order discontinuity at the moment which is advantageous for this, and to perform a small, additional "smoothing" of the region adjacent to it.

9. DISCONTINUITIES (GENERAL PROPERTIES) /66

The initial discontinuity of an expansion wave, which was investigated in the preceding section, exists only for one instant. However, as is well known from hydrodynamics, more stable discontinuities existing for a long period of time may appear in an ideal, compressible liquid. The discontinuity point (corresponding in space to the discontinuity surface, which is flat, cylindrical, or spherical in our case) moving along the r, t-plane describes a line - the discontinuity line. When passing through this line, the hydrodynamic quantities (u, p, et cetera, but not r, t, R) perform jumps. The limiting values of these quantities on both sides of the discontinuity fulfill definite conditions which we shall now briefly derive.

Let AB be the discontinuity line. It divides the r, t-plane into two parts, which we shall designate by the indices 1 and 2 (Figure 20). Let us enclose an integral on this line by a closed shape having the form of a narrow band; let the shape intersect the discontinuity line at the points C and D.

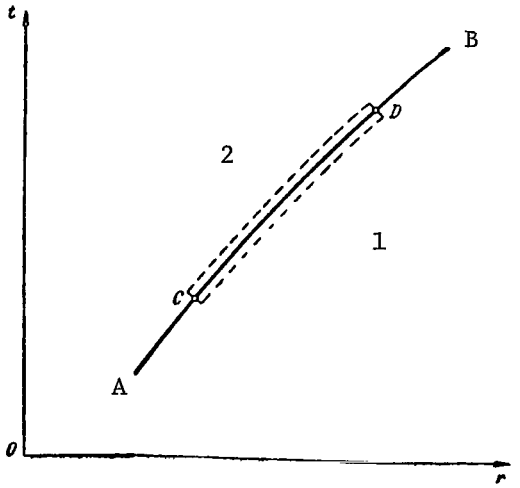


Figure 20

The integral laws of conservation (1.19) are valid for any closed shape, independently of whether the integrands are continuous. In particular, we may apply them to our shape. We shall decrease the width of our band, keeping the points C and D stationary; the shape joins the C segment of the discontinuity line at the limit. Let us determine what the laws of conservation (1.19) provide us with at the limit.

First of all, we may discard the parts corresponding to integration /67 over an infinitely small "transverse" contour interval containing the points

C and D in the contour integrals on the left. Only two linear integrals remain - from point C to point D in region 1, and from point D to point C in region 2. In addition, since the band area vanishes, the integral over the surface disappears in the right part of the law of conservation of momentum [second equation (1.19)]. All three laws of conservation assume the following form

$$\int_C^D f_1\,dr - g_1\,dt + \int_D^C f_2\,dr - g_2\,dt = 0. \tag{9.1}$$

The indices 1 and 2 indicate in what region (i.e., at which side of the discontinuity line) the values of the integrands must be selected. Both integrals are selected along the CD interval of the discontinuity line.

Relationship (9.1) can be rewritten as follows:

$$\int_C^D f_1\,dr - g_1\,dt = \int_C^D f_2\,dr - g_2\,dt. \tag{9.2}$$

Let the equation for the discontinuity line be $\frac{dr}{dt} = D$ so that D is the velocity of the discontinuity surface in space. Then equation (9.2) assumes the following form

$$\int_C^D (f_1 D - g_1)\,dt = \int_C^D (f_2 D - g_2)\,dt. \tag{9.3}$$

However, since the CD interval may be completely arbitrary, the following equation for the integrands follows from the equation for the integrals (9.3):

$$f_1 D - g_1 = f_2 D - g_2. \tag{9.3'}$$

This is the desired relationship on the discontinuity line. We must now only substitute the specific expressions for the integrands from the laws of conservation (1.19) in (9.3'). instead of f and g. Thus, the factors r^ν are eliminated, and we obtain three relationships:

$$\left.\begin{aligned} \rho_1 D - \rho_1 u_1 &= \rho_2 D - \rho_2 u_2, \\ \rho_1 u_1 D - (\rho_1 u_1^2 + p_1) &= \rho_2 u_2 D - (\rho_2 u_2^2 + p_2), \\ \rho_1\left(\varepsilon_1 + \frac{u_1^2}{2}\right) D - u_1\left[\rho_1\left(\varepsilon_1 + \frac{u_1^2}{2}\right) + p_1\right] &= \\ = \rho_2\left(\varepsilon_2 + \frac{u_2^2}{2}\right) D - u_2\left[\rho_2\left(\varepsilon_2 + \frac{u_2^2}{2}\right) + p_2\right], & \end{aligned}\right\} \tag{9.4}$$

which can be rewritten as: /68

$$\left.\begin{aligned} \rho_1(D-u_1)-\rho_2(D-u_2)&=0,\\ \rho_1u_1(D-u_1)-\rho_2u_2(D-u_2)&=p_1-p_2,\\ \rho_1\left(\varepsilon_1+\frac{u_1^2}{2}\right)(D-u_1)-\rho_2\left(\varepsilon_2+\frac{u_2^2}{2}\right)(D-u_2)&=u_1p_1-u_2p_2. \end{aligned}\right\} \qquad (9.5)$$

The discontinuities may be divided into two types which are different qualitatively. The first type includes discontinuities through whose surface there is no flux of matter; these discontinuities are called contact discontinuities. It can be readily seen that this condition may be written in the following form

$$D=u,$$

and we may use both u_1 and u_2 for u, because the first equation (9.5) shows that the conditions $D = u_1$ and $D = u_2$ are equivalent. It thus follows directly that

$$u_1=u_2.$$

In addition, we obtain the following from $D - u_1 = D - u_2 = 0$, according to the second equation (9.5):

$$p_1=p_2.$$

The third law of conservation does not provide anything new.

Thus, the contact discontinuity may be characterized by the following relationships:

$$\left.\begin{aligned} D&=u,\\ u_1&=u_2,\\ p_1&=p_2. \end{aligned}\right\} \qquad (9.6)$$

On the r, t-plane, the contact discontinuity line represents an ordinary trajectory; the particle velocity u and the pressure p remain constant when passing through this line. The density ρ and the entropy s undergo a discontinuity. The surface of the contact discontinuity in space divides substances in different thermodynamic states - for example, having a different temperature - into two parts. In view of the fact that there is no thermoconductivity, this temperature difference cannot be equalized.

Surfaces dividing different substances must pertain to contact discontinuities. In this case, it may be stated that the equation of state changes on the r, t plane on the contact discontinuity line. The relationships (9.6) always remain in force.

We would like to point out the obvious fact that the contact discontinuity line on the r, t plane is always timelike.

Discontinuities of the second type - those through which there is a /69 flux of matter - are called shock waves. Relationships (9.5) cannot be simplified for them, as was the case for contact discontinuities. However, we can transform these relationships. We may exclude $D - u_1$ and $D - u_2$ from them.

First of all, solving the first two equations with respect to $D - u_1$ and $D - u_2$ we obtain

$$\left.\begin{aligned} D-u_1 &= \frac{p_2-p_1}{\rho_1(u_2-u_1)}, \\ D-u_2 &= \frac{p_2-p_1}{\rho_2(u_2-u_1)} \end{aligned}\right\} \tag{9.7}$$

and, in addition

$$\rho_1(D-u_1) = \rho_2(D-u_2) = \frac{p_2-p_1}{u_2-u_1}. \tag{9.8}$$

Subtracting the second equation from the first (9.7), we have

$$u_2-u_1 = \frac{p_2-p_1}{u_2-u_1}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right),$$

or

$$(u_2-u_1)^2 = -(p_2-p_1)\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right). \tag{9.9}$$

Substituting the quantities $\rho_1(D - u_1)$ and $\rho_2(D - u_2)$ from (9.8) in the third equation (9.5), we obtain the following (changing signs in both parts)

$$\frac{p_2-p_1}{u_2-u_1}\left(\varepsilon_2-\varepsilon_1+\frac{u_2^2-u_1^2}{2}\right) = u_2p_2-u_1p_1.$$

As may be readily seen, the right side may be written in the following form

$$\frac{(u_1+u_2)(p_2-p_1)+(p_1+p_2)(u_2-u_1)}{2}.$$

Removing the brackets in the left side, we obtain

$$\frac{p_2-p_1}{u_2-u_1}(\varepsilon_2-\varepsilon_1)+\frac{(u_1+u_2)(p_2-p_1)}{2} = \frac{(u_1+u_2)(p_2-p_1)+(p_1+p_2)(u_2-u_1)}{2}.$$

or, after contraction,

$$\varepsilon_2 - \varepsilon_1 = \frac{p_1 + p_2}{2} \cdot \frac{(u_2 - u_1)^2}{p_2 - p_1}.$$

The second factor in the right side is $-\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)$ according to (9.9). Therefore, we have

$$\varepsilon_2 - \varepsilon_1 = -\frac{p_1 + p_2}{2}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right). \tag{9.10}$$

It follows from (9.9) that /70

$$\frac{p_2 - p_1}{u_2 - u_1} = -\frac{u_2 - u_1}{\frac{1}{\rho_2} - \frac{1}{\rho_1}} = \rho_1\rho_2\frac{u_2 - u_1}{\rho_2 - \rho_1}.$$

Substituting this in (9.7), we have

$$D - u_1 = \frac{\rho_2}{\rho_2 - \rho_1}(u_2 - u_1). \tag{9.11}$$

Let us introduce the following notation

$$\sigma = \frac{\rho_2}{\rho_1};$$

where σ is the <u>compression</u> through the shock wave. Then (9.11) may be rewritten in the form

$$D - u_1 = \frac{\sigma}{\sigma - 1}(u_2 - u_1). \tag{9.12}$$

Thus, on the basis of (9.9), (9.10) and (9.11) we can write the relationships for the shock wave in the following form

$$\left.\begin{aligned} D - u_1 &= \frac{\rho_2}{\rho_2 - \rho_1}(u_2 - u_1), \\ (u_2 - u_1)^2 &= -(p_2 - p_1)\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right), \\ \varepsilon_2 - \varepsilon_1 &= -\frac{p_1 + p_2}{2}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right). \end{aligned}\right\} \tag{9.13}$$

Up to this point, the indices 1 and 2 were applied to regions lying on both sides of the shock wave, in a completely arbitrary manner. We shall now

stipulate that the index 1 designates the region from which the flux of matter is directed through the wave, and the index 2 designates the region to which this flux is directed. The Lagrangian coordinate R along the shock wave front must change. If it increases (with an increase in t), in the cylindrical and spherical cases this will be a diverging wave; if R decreases, the wave will be a converging wave. For purposes of uniformity, we shall use these terms (diverging and converging waves) in the flat case. For the diverging waves, the region lying to the right of the discontinuity line (on the r, t plane) is designated by the index 1, according to our definition; for the converging waves, the index 1 designates the region to the left of the discontinuity.

Let us write the equation of state for matter in the following form

$$\rho = \rho(p, s).$$

When an investigation of shock waves in matter is performed in hydrodynamics, the following condition is usually imposed

$$\frac{\partial^2}{\partial p^2}\left(\frac{1}{\rho}\right) > 0. \tag{9.14}$$

We shall also adhere to this stipulation. The equations of state which /71
we discussed above satisfy condition (9.14).

The following properties of shock waves are presented in courses on hydrodynamics under the condition of (9.14):

1. Shock waves are always compression waves, i.e., the following inequalities are always valid

$$p_2 > p_1, \ \rho_2 > \rho_1.$$

(The so-called Cemplen theorem).

2. The following double inequality is fulfilled for diverging shock waves

$$\alpha_1 > D > \alpha_2,$$

and for converging waves the following is fulfilled

$$\beta_1 < D < \beta_2.$$

3. For an ideal gas, and also for all the equations of state investigated previously, the ratio $\frac{\rho_2}{\rho_1}$ is limited:

$$\frac{\rho_2}{\rho_1} < h = \frac{\varkappa + 1}{\varkappa - 1},$$

while the ratio $\frac{p_2}{p_1}$ may be as large as desired.

4. The quantity u is always smaller to the right of the wave front (on the r, t plane) than it is to the left of the front (both for converging and for diverging waves).

Together with equations (9.13), the first characteristic entails the following inequality for all our substances

$$c_2 > c_1,\ z_2 > z_1,\ v_2 > v_1.$$

The entropy must increase on the shock wave front.

The second characteristic indicates the position of the shock wave front line on the r, t-plane with respect to the grid of the characteristics. For example, for a diverging wave this position will correspond to that shown in Figure 21. As the front of the shock wave "catches up," it "truncates" the α-characteristics of region 1. The α-characteristics of region 2 "catch up with" the shock wave in their turn. Thus, the velocity of the shock wave front will be supersonic for region 1, and subsonic for region 2. It may also be stated that the line of the shock wave front for region 1 is spacelike, and for region 2 it is timelike.

Let us turn once more to the relationships (9.13). The second equation /72 relates the square of the velocity change $u_2 - u_1$ with the pressures and densities at the shock wave. If we want to calculate $u_2 - u_1$ on the basis of this equation (assuming that p_1, p_2, ρ_1, ρ_2 are given), when the square root is

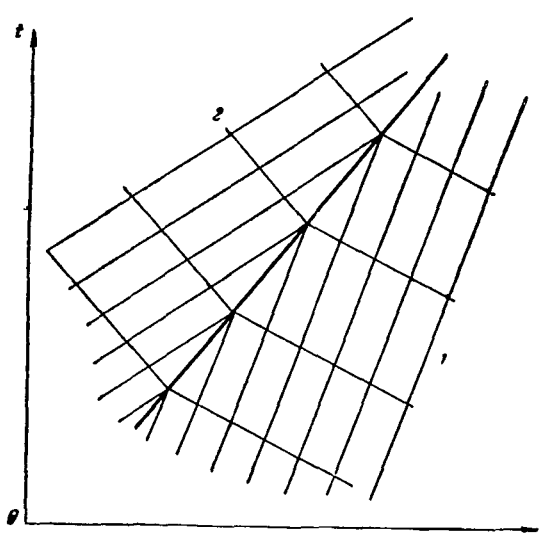


Figure 21

taken on the right side, it would be necessary to attribute a specific sign to it. The fourth of the shock wave characteristics enumerated above indicates that it would be necessary to choose a plus sign for diverging waves, and a minus sign for converging waves.

The third relationship (9.13) is unusual, due to the fact that it contains only thermodynamic quantities - pressure, density, and internal energy. It is called the Hugoniot adiabatic equation. We should point out that, as shown in hydrodynamics, there is no combination of thermodynamic quantities which could remain unchanged when passing through the shock wave front.

10. CONTACT DISCONTINUITIES /73

In its broad outlines, the procedure to be followed in computing the contact discontinuity is as follows. Let us assume that point 1 of the discontinuity (Figure 22) has already been computed - i.e., both its coordinates and all the quantities on both sides of the discontinuity are known. Using the usual procedure, we may calculate both characteristics emanating from point 1 - the α-characteristic 1 - 1' and the β-characteristic 1 - 1". Let us draw the line $\frac{dr}{dt} = u_1$ from point 1. It will approximately represent the contact discontinuity line. Let us select points 2 and 3 on the 1 - 1' and 1 - 1" characteristics in such a way that the α-characteristic emanating from point 2, and the β-characteristic emanating from point 3, intersect the discontinuity line at one and the same point 4. This will be the desired subsequent point of the contact discontinuity.

We must compute six quantities at point 4 - for example, the value of v, A, B on both sides of the discontinuity. In order to do this, we must have six equations. We have two equations from (9.6) - this is the condition for the velocities u and the pressures p to be equal on both sides of the discontinuity. In addition, the Lagrangian coordinate R, and consequently the entropy s, are retained along the contact discontinuity line. The entropy has two different values (generally speaking) on both sides of the discontinuity; in particular, each of them is shifted from point 1 to point 4. This provides us with two equations. Finally, we can obtain the last two equations from the relationships for the 2 - 4 and 3 - 4 characteristics.

Specific variants of the computational formulas can be different. Let us analyze the two main variants, the first of which is the simplest and has universal application in a certain sense. Let us first present certain stipulations.

Each point lying on the discontinuity line contains two "sets" of quantities pertaining to two sides of the discontinuity. Therefore, it is advantageous to assume that it has not one, but two points, whose coordinates t, r, R coincide. In accordance with this, we shall change the notation in Figure 22, and shall employ different notation, indicated in Figure 23. We shall /74 use the index 1 to designate the region located to the left of the discontinuity; we shall use the index 2 to designate the region on the right. We shall employ these indices to designate the constants ($\varkappa$,ρ_0' , c_0, et cetera) pertaining to these regions.

Let us assume that the coordinates of points 5 and 6 have been found. We can now find v at these points from $v_5 = v_1$, $v_6 = v_2$. Let us now determine p and u at points 5 and 6. Since $p_5 = p_6$, $u_5 = u_6$, we actually need to find two, and not four, quantities. In order to do this, it is simplest to employ the equations of characteristics in the form (2.3):

$$\frac{dp}{\rho c} \pm du = F\,dt.$$

As is customary, let us replace the differentials by the finite differences, and we obtain the two equations

$$\frac{p_5 - p_3}{\rho_3 c_3} + (u_5 - u_3) = F_3 (t_5 - t_3),$$

$$\frac{p_6 - p_4}{\rho_4 c_4} - (u_6 - u_4) = F_4 (t_6 - t_4).$$

Solving them with respect to $p_5 = p_6$ and $u_5 = u_6$, we can compute the quantities desired.

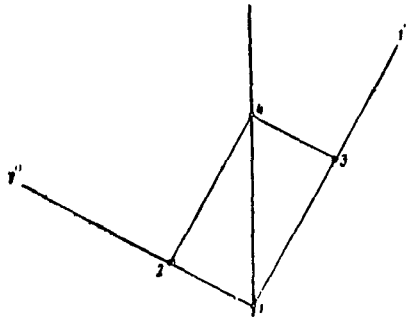


Figure 22

Points 3 and 4 are located on fixed characteristics; the coordinates of points 5 and 6 coincide. The directions of segments 3 - 5 and 4 - 6 are also fully determined. It follows from this that we have one degree of freedom at our disposal when constructing a "figure" consisting of points 1, 2, 3, 4, 5, 6. Assigning one of the coordinates to one of the points 3 - 6, we may determine the entire "figure". Which coordinate must be disposed of? /75

In the overwhelming majority of cases, it is most convenient to select one of the points 3, 4. We may select one of the points already computed corresponding to the characteristic, which saves us from unnecessary interpolation. As a rule, we should select the point which is closest to point 1 or point 2. In view of the special circumstances, only in individual cases is it necessary for us, for example, to note the coordinates of points 5, 6.

For purposes of definition, we shall assume that we have selected point 4. We can obtain the coordinates of points 5, 6 (in the first approximation) by finding the point at which the line $dr = u_1\,dt$ emanating from point 1 (or 2) intersects the line $dr = \beta_4\,dt$ emanating from point 4. We must now determine point 3.

It is necessary to know the direction of the 3 - 5 characteristic in order to do this. However, we still do not know α_3, or α_5. We must take α from any point which is close to point 3. This point may be readily found

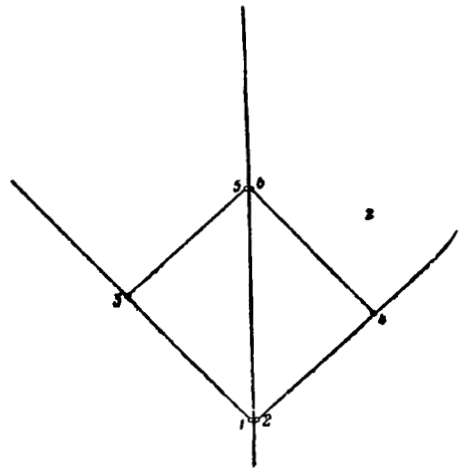


Figure 23

from the r, t graph, on which a grid showing the characteristics for the computed portion of region 1 must be compiled at the time under consideration. By plotting points 5, 6 on this graph, we can determine where point 3 falls with sufficient accuracy. It is assumed in the computational formulas presented below that point 1 is closest to point 3. This fact must be kept in mind during the practical application of these formulas.

Drawing the lines $dr = \beta_1 dt$ through point 1 and $dr = \alpha_1 dt$ through point 5, and finding their point of intersection, we may determine the coordinates of point 3. We may find all of the quantities which we need at point 3 by interpolation along the β-characteristic emanating from point 1.

On the basis of the statements presented above, we may now compute points 5 and 6, after which we shall perform the recalculation. This is done in the usual manner by replacing the coefficients of the differentials by the corresponding mean coefficients in all the formulas.

By way of an example, let us present a variation of the specific computational formulas. It is assumed that both substances adjacent to the discontinuity line are ideal gases. Let us introduce the following notation

$$g = \frac{1}{\rho c} = \frac{1}{a^2} \frac{v}{z^h}.$$

The substitutions which are made during the recalculation are shown in the formulas on the right in parentheses. The arithmetic means are designated by the double indices - for example, $u_{26} = \frac{1}{2}(u_2 + u_6)$:

$$t_5 = t_6 = \frac{\beta_4 t_4 - u_2 t_2 + r_2 - r_4}{\beta_4 - u_2}, \qquad (u_2 \to u_{26},\ \beta_4 \to \beta_{46})$$

$$r_5 = r_6 = r_2 + u_2(t_6 - t_2) = r_4 + \beta_4(t_6 - t_4), \quad (u_2 \to u_{26},\ \beta_4 \to \beta_{46})$$

$$t_3 = \frac{\beta_1 t_1 - \alpha_1 t_5 + r_5 - r_1}{\beta_1 - \alpha_1}. \qquad (\beta_1 \to \beta_{13},\ \alpha_1 \to \alpha_{35})$$

/76

Interpolation of point 3:

$$p_3 = \frac{a_1^2}{\varkappa_1} z_3^{h_1+1}, \quad p_4 = \frac{a_2^2}{\varkappa_2} z_4^{h_2+1},$$

$$g_3 = \frac{1}{a_1^2}\frac{v_3}{z_3^{h_1}} \qquad g_4 = \frac{1}{a_2^2}\frac{v_4}{z_6^{h_2}},$$

$$F_3 = -\nu\frac{u_3 v_3 z_3}{r_3}, \quad F_4 = -\nu\frac{u_4 v_4 z_4}{r_4},$$

$$p_5 = p_6 = \frac{g_3 p_3 + g_4 p_4 + u_3 - u_4 + F_3(t_5 - t_3) + F_4(t_6 - t_4)}{g_3 + g_4},$$

$$u_5 = u_6 = u_3 + F_3(t_5 - t_3) - g_3(p_5 - p_3) = \quad \begin{pmatrix} g_3 \to g_{35} \\ g_4 \to g_{46} \\ F_3 \to F_{35} \\ F_4 \to F_{46} \end{pmatrix}$$

$$= u_4 - F_4(t_6 - t_4) + g_4(p_6 - p_4),$$

$$v_5 = v_1, \quad v_6 = v_2,$$

$$z_5 = \left(\frac{\varkappa_1}{a_1^2} p_5\right)^{\frac{1}{h_1+1}}, \qquad z_6 = \left(\frac{\varkappa_2}{a_2^2} p_6\right)^{\frac{1}{h_2+1}},$$

$$\alpha_5 = u_5 + v_5 z_5, \quad \beta_6 = u_6 - v_6 z_6,$$

$$F_5 = -\nu\frac{u_5 v_5 z_5}{r_5}, \qquad F_6 = -\nu\frac{u_5 v_6 z_6}{r_6},$$

$$g_5 = \frac{1}{a_1^2}\frac{v_5}{z_5^{h_1}}, \qquad g_6 = \frac{1}{a_2^2}\frac{v_6}{z_6^{h_2}}.$$

(10.1)

Similarly to v, R can be simply transposed along the discontinuity line, so that the following relationships must be added to formulas (10.1):

$$R_5 = R_1, \quad R_6 = R_2.$$

As a rule, $R_1 = R_2$ and $R_5 = R_6$. However, for purposes of control it is recommended that R be integrated along the segments of the 3 - 5 and 4 - 6

characteristics. This may be performed when the recalculation is finished, substituting the mean values of R' [Equation (7.7)] in the integration formula. We shall return to the problem of controlling R on contact discontinuities when we discuss general integral control of the solution.

When calculating each point of the contact discontinuity, if we select the calculated point closest to point 2 as point 4, then the grid for the characteristics assumes a form which is approximately the same as that shown in Figure 24. If we start with the selection of point 5, then the pic- /77 ture is relatively symmetrical. As we may see, the characteristics of one set (β in Figure 24) "pass" through the discontinuity line and "are reflected" from it. The characteristics of the other set are "truncated" at the discontinuity

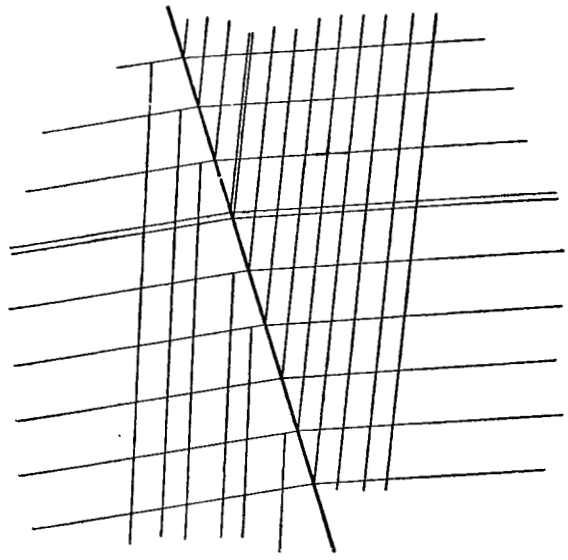


Figure 24

line. This factor must be taken into account when a first-order discontinuity approaches a contact discontinuity. If, as is shown in Figure 24, the line of the first-order discontinuity (the double line in the figure) represents the characteristic which is "passed" through the contact discontinuity and "reflected" from it, the first-order discontinuity introduces no additional difficulties. In the opposite case, when the first-order discontinuity closely approaches the contact discontinuity line, interpolation of point 3 (see Figure 23) is impossible. We must thus change the computational order, selecting the first-order discontinuity point as point 3.

In spite of all the advantages of this method, it is not always advantageous to employ it in practice. The quantity p frequently changes from point to point so greatly, that direct integration of equations (2.3) entails a great amount of error, as well as the necessity of decreasing the step close to the

discontinuity. When the discontinuity points are computed, it is thus desirable to integrate equations (2.5) - i.e., to return to Reimann invariants. It is true that the ratio $p_5 = p_6$ is replaced by a more complex one, but this is partially offset by other advantages. We shall now briefly sketch the second method for computing the contact discontinuity, without presenting the final computational formulas. /78

It can be seen from Figure 23 that we may find A_5 and B_6 by performing integration along the segments of the 3 - 5 and 4 - 6 characteristics. After this process, since we may assume that v_5 and v_6 are known, we may employ equations $p_5 = p_6$ and $u_5 = u_6$ to compute B_5 and A_6. This may be done as follows:

Since $u_5 = u_6$, from the formulas

$$A_5 = (h_1 - 1)c_5 + u_5,$$
$$B_6 = (h_2 - 1)c_6 - u_6$$

it follows that

$$A_5 + B_6 = (h_1 - 1)c_5 + (h_2 - 1)c_6. \tag{10.2}$$

In addition, since $p_5 = p_6$,

$$\frac{a_1^2}{\varkappa_1} z_5^{h_1+1} = \frac{a_2^2}{\varkappa_2} z_6^{h_2+1},$$

or, since $z = \frac{c}{v}$

$$\frac{a_1^2}{\varkappa_1 v_5^{h_1+1}} c_5^{h_1+1} = \frac{a_2^2}{\varkappa_2 v_6^{h_2+1}} c_6^{h_2+1}. \tag{10.3}$$

The quantities v_5 and v_6 may be assumed to be constant. If we now solve equations (10.2) and (10.3) with respect to c_5 and c_6, we find that both c_5 and c_6 are functions of $A_5 + B_6$. Consequently, after integration of A_5 and B_6, we may immediately determine c_5 and c_6, as well as

$$B_5 = 2(h_1 - 1)c_5 - A_5,$$
$$A_6 = 2(h_2 - 1)c_6 - B_6.$$

Here the difficulty is encountered in equation (10.3), which is nonlinear. However, we may proceed as follows. For example, let us assume a certain sequence in the values of c_5. The corresponding sequence of the values for c_6 may be readily computed from equation (10.3), and the sequence of $A_5 + B_6$ may

be computed from (10.2). As a result we obtain a table containing three columns for the quantities c_5, c_6 and $A_5 + B_6$. If we know $A_5 + B_6$, we may find both c_5 and c_6 from this table. For greater convenience, this table may be first pre-interpolated, so that the values of $A_5 + B_6$ are equidistant.

Unfortunately, such a table cannot have universal application, since it depends on v_5 and v_6. Therefore, it must be compiled again for each section of the contact discontinuity with the data of v_5 and v_6. However, as a rule, this additional work is worth the effort.

The introduction of Reimann invariants makes it possible to take one step in simplifying the calculation - to exclude the interpolation of point 3 (or 4) (Figure 23). Let us select points which have already been computed as /79 points 3 and 4. Then the coordinates of points 5 and 6 will not coincide. For purposes of definition, let us assume that point 5 lies above point 6 (Figure 25). By finding the coordinates of these points we may compute A_5 and B_6 in the usual way. By employing points 5, 1 and the "left" point calculated previously (lying above point 1) as interpolation points, we may interpolate the

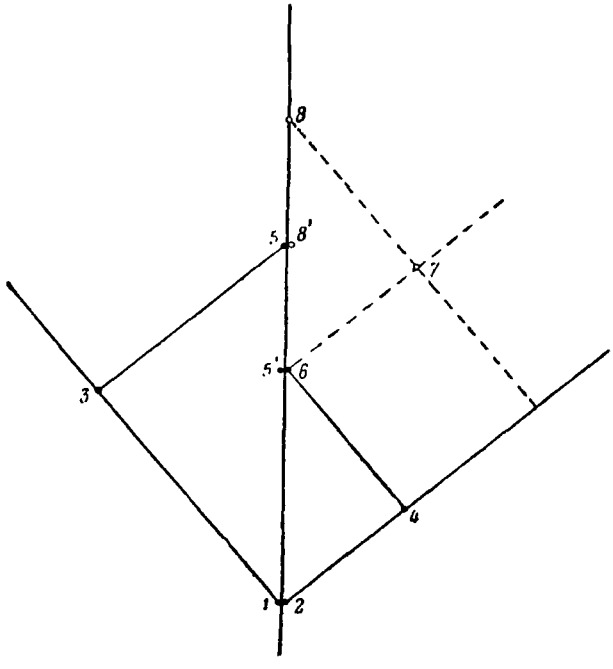


Figure 25

"left" quantity A at point 5', whose coordinates coincide with the coordinates of point 6. As indicated above, by knowing A_5 and B_6 we may compute point 6. Point 5' is an auxiliary point, and it may be computed with no difficulty, as a rule.

The recalculation entails certain difficulties - namely, the recalculation of A_5. Except for A, at point 5 nothing is unknown to us, and in order to recalculate A it is necessary to know α, F and z. The following procedure may be assumed. After the first approximation of point 6 is performed, we may extrapolate u at point 5, on the basis of points 6, 2 and the point previously computed. Knowing A_5 and u_5, we may readily calculate α_5, F_5, z_5; we may then recalculate A_5. With A_5 defined more accurately, we may recalculate point 6.

The calculation of point 5 will be as follows. At this time we already know t_5, r_5, A_5, u_5, α_5, F_5, Z_5, which have been computed in the customary way by extrapolation of u. However, nothing prevents us from taking them for the first approximation. It only remains to recalculate point 5. In order to do this, it is necessary to know B_8 (see Figure 25), which we may find by integration along the 7-8 segment. Let us then extrapolate u at point 8, let us determine B_8 more accurately, let us interopolate it at point 8' and finally let us recalculate point 5. /80

The rest of the computational procedure is now clear. As we may see, the "process" for this computation differs considerably from the "Euler method with recalculation," but the over-all order of accuracy remains as before. The

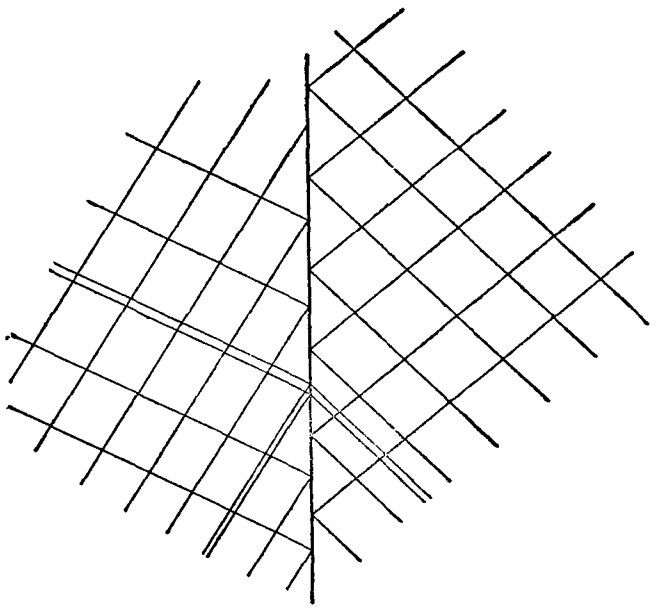


Figure 26

network of characteristics will have the form shown in Figure 26, so that the characteristics of both sets are "reflected" from the discontinuity line. When necessary, nothing prevents us from "passing" a certain characteristic along the discontinuity, since after this all of the quantities at point 5' may be

readily computed, since point 6 (Figure 25) has been calculated. After this, it is possible to "release" the next β-characteristic from point 5'. Such a procedure may be necessary, for example, when a characteristic supporting a first-order discontinuity (the double line in Figure 26) reaches the line of the contact discontinuity.

Both methods presented for computing the contact discontinuity have their advantages and disadvantages. The first method entails the necessity of interpolation along the characteristics, but it has great logical simplicity and computational uniformity at each step. Therefore, it is more suitable for computations on computers. The second method is preferable for computations "by hand," since it is more economical and flexible.

Figure 21 illustrates the position of the line of the shock wave front with respect to the characteristics. In contrast to the contact discontinuity, there is no symmetry between the regions lying on both sides of the discontinuity. If it is reasonable to speak of the "right" and "left" regions in the case of the contact discontinuity, then we shall employ a different terminology for the shock wave. We shall call region 1 the lower region, and region 2 the upper region. Correspondingly, at each point on the shock wave front we may speak of lower and upper values for each quantity.

For the lower region, the line of the shock wave front is a spacelike line. Therefore, nothing prevents us from solving equations (2.1) continuously in the "upper" region, ignoring the discontinuity. In physical terms, this "upper" region is purely fictitious; it reflects the motion of a substance which would exist if there were no shock waves. Naturally, every effort must be made to see that this "excessive" computation is reduced to a minimum. However, during a numerical computation such a continuation of the lower region beyond the "prescribed" boundaries must always be performed. From this point on, we shall always assume that this continuation of the lower region has already been effected, and our problem consists of "truncating" the portion of the characteristics field which was computed previously, in order to "pass" the shock wave along this "prepared" region.

Let us assume that point 1 of the shock wave front (Figure 27) has already been computed - i.e., both the lower and upper values of all the quantities are known at this point. We may employ the first formula (9.13) to calculate the velocity D of the wave front at the point 1. The line $dr = D_1\,dt$ drawn through point 1 approximately expresses the discontinuity line. On this line, let us select a certain point 2 which we shall intend as the next point of the wave front.

The characteristic 1 - 1' emanates from point 1 upward (since the discontinuity line is timelike for the upper region, and only one characteristic may emanate upward). We may assume that this characteristic is already known. Let us select point 3 upon it, in such a way that the characteristic of another set (with respect to 1 - 1') from point 3 falls exactly at point 2. /82

Since the line 1 - 2 passes by definition along the lower region which has already been computed, we may find the lower quantities at point 2 directly. We already know the coordinates of point 2. We must now compute any three quantities - for example, u, p, ρ - at point 2. We must have three equations in order to do this. Two equations provide us with relationship (9.13)[the second and third . We should point out that the index 1 in these equations does not designate the quantities at point 1 (Figure 27), but the lower values at point 2]. The third equation provides us with the relationship along the 3 - 2 characteristic, rewritten with finite differences. In solving this system, we may compute point 2 entirely.

Thus, the calculation of the shock wave front point may be divided into the following four steps:

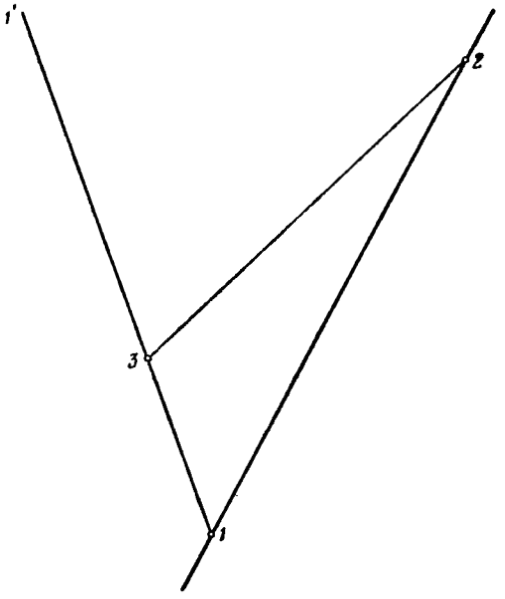


Figure 27

(1) Calculation of the coordinates of the new point;

(2) Determination of the lower quantities at this point;

(3) Drawing of the "overtaking" characteristic (3 - 2 characteristic in Figure 27);

(4) Solution of the system of equations at the wave front.

There is a close correlation between points 1, 2, and 3; point 4 is independent to a significant extent. Therefore, we shall first examine the first three steps in the calculation, postponing the last step until later.

Point 2 entails the primary difficulty, since we must deal here with the interpolation of functions of two variables. We know the values of all quantities in the lower region at the points in the characteristics network, and if we select point 2 (Figure 27) arbitrarily, it will fall within the "parallelograms of the characteristics." Interpolation (quadratic!) of the functions of two variables is a very cumbersome problem, and our goal is to reduce it to interpolation over one variable, i.e., to interpolation along a a certain line.

One possible solution becomes immediately apparent. We must select point 2 in such a way that it lies on one of the characteristics in the lower region

(see Figure 28, where these characteristics are designated by a dashed line). The computational procedure may be as follows (for purposes of definition, a diverging wave is examined): /83

1. Let us draw the line $dr = D_1\,dt$ from point 1, and let us draw the line $dr = \beta_1\,dt$ from point 4 in the lower region. Let us determine the point at which they intersect, which will be point 2.

2. Performing interpolation along the "lower" 4 - 5 characteristic, we may find the lower quantities at point 2.

3. Let us draw the line $dr = \beta_1\,dt$ from point 1, and the line $dr = \alpha_1\,dt$ from point 2. The point at which they intersect will be point 3.

4. Performing interpolation along the 1 - 1' characteristic, we may find all of the quantities at point 3.

5. Let us solve the system of equations, let us compute all of the upper quantities at point 2, and let us perform a recalculation.

We have assumed that point 1 is closest to point 3 on the 1 - 1' characteristic. If we find a closer point, it would be advantageous to select it instead of point 1, in step 3. The quantity $\frac{\alpha_2 + \alpha_3}{2}$ is selected during the recalculation, instead of α_1 (the reader may compare this step with the determination of point 3 in the calculation of the contact discontinuity; see Figure 22 and the text pertaining to it).

Almost all of the statements presented in Section 10 regarding the first method for calculating the contact discontinuity may be repeated with respect to this method. It is quite simple logically, but entails the necessity of "inserting" point 3. Therefore, we shall investigate another method, where this "insertion" is not requisite.

Let us assume that point 1 of the shock wave (Figure 29) has been computed. On the characteristic emanating from this point, let us select point 3 which has also been computed. Let us draw the line $dr = D_1\,dt$ from point 1, and the line $dr = \alpha_3\,dt$ from point 3. Let these lines intersect at point 2. In addition, let us have the line $dr = D_1\,dt$ intersect with the "lower" characteristic passing ahead of point 2. We may obtain the auxiliary point 4, at which we may find the lower quantities by interpolation along the characteristics.

Taking point 4, and also the auxiliary points 5 and 6 which have already been computed, we may interpolate the lower quantities at point 2 over them. After this, we may solve the system of equations and may compute point 2.

During the recalculation, it must be verified that the auxiliary point is "placed" as accurately as possible on the line of the shock wave front. In order to do this, we must first extrapolate D at point 4 along points 1 and 2 and,

when the coordinates of point 4 are determined more accurately, we must draw the line $dr = \frac{1}{2}(D_1 + D_4)\,dt$ from point 1. The rest of the recalculation process is quite clear, and requires no additional clarification.

The presence of first-order discontinuities entails no additional difficulties. If a first-order discontinuity is contained in the "lower" data, then when it approaches the line of the wave front, the first method must be employed, taking the next wave point on the characteristic supporting the first-order discontinuity. If the first-order discontinuity "overtakes" the shock wave in the upper region, we shall employ the second method, taking the point of the first-order discontinuity as point 3. In both cases, the characteristic supporting the first-order discontinuity passes upward from the new point 2 of the shock wave. /84

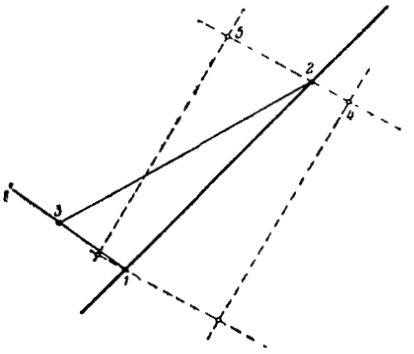


Figure 28

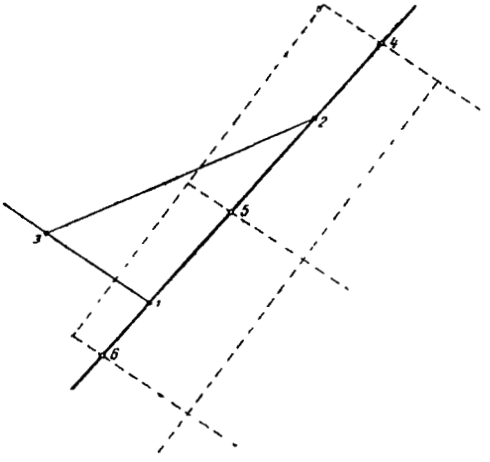


Figure 29

Let us now turn to the problem of solving the system of equations on the shock wave front; let us only analyze the case of an ideal gas. Two methods may also be proposed here.

As is known, the internal energy ε of an ideal gas is expressed by its density ρ and pressure p in the following way

$$\varepsilon = \frac{1}{\varkappa - 1} \frac{p}{\rho}. \tag{11.1}$$

Substituting this expression in the third equation (9.13), we obtain the adiabatic equation of Hugoniot in a form which is given in each course on hydrodynamics:

$$\frac{\rho_2}{\rho_1} = \frac{hp_2 + p_1}{p_2 + hp_1}. \tag{11.2}$$

In passing, we might mention that it follows from this equation that the ratio $\frac{\rho_2}{\rho_1}$ across the shock wave cannot be greater than h. In actuality, for a constant p_1 and increasing p_2 the right part of formula (11.2) increases monotonically, and strives to h.

Let us now express ρ and p in (11.2) by z and v, according to (5.11). We obtain /85

$$\left(\frac{z_2}{z_1}\right)^{h-1}\left(\frac{v_1}{v_2}\right)^2 = \frac{hz_2^{h+1} + z_1^{h+1}}{z_2^{h+1} + hz_1^{h+1}},$$

or, assuming that $\tau = \frac{z_2}{z_1}$

$$\left(\frac{v_2}{v_1}\right)^2 = \tau^{h-1}\frac{\tau^{h+1} + h}{h\tau^{h+1} + 1}. \tag{11.3}$$

In the right part of the second equation, let us remove $\frac{p_1}{\rho_1}$ from the parentheses:

$$(u_2 - u_1)^2 = -\frac{p_1}{\rho_1}\left(\frac{p_2}{p_1} - 1\right)\left(\frac{\rho_1}{\rho_2} - 1\right).$$

Instead of $\frac{\rho_2}{\rho_1}$, let us substitute its expression from (11.2), and let us note that $\frac{p_2}{p_1} = \tau^h + 1$, $\frac{p_1}{\rho_1} = \frac{1}{\varkappa} c_1^2$. Then, after several elementary computations, we

obtain

$$(u_2 - u_1)^2 = \frac{h-1}{\varkappa} c_1^2 \frac{(\tau^{h+1}-1)^2}{h\tau^{h+1}+1}. \tag{11.4}$$

Finally, substituting $\frac{\rho_2}{\rho_1}$ from (11.2) and $u_2 - u_1$ from (11.4), in the first relationship (9.13), we obtain

$$(D - u_1)^2 = \frac{c_1^2}{h+1}(h\tau^{h+1}+1). \tag{11.5}$$

Let us collect the transformed equations (11.3), (11.4) and (11.5) together:

$$\left.\begin{aligned} (u_2 - u_1)^2 &= \frac{h-1}{\varkappa} c_1^2 \frac{(\tau^{h+1}-1)^2}{h\tau^{h+1}+1}, \\ \left(\frac{v_2}{v_1}\right)^2 &= \tau^{h-1}\frac{\tau^{h+1}+h}{h\tau^{h+1}+1}, \\ (D - u_1)^2 &= \frac{c_1^2}{h+1}(h\tau^{h+1}+1). \end{aligned}\right\} \tag{11.6}$$

When extracting the square root in the first and third equations, we must allow for the shock wave direction - for diverging waves we must select the plus sign ($u_2 > u_1$, $D > u_1$); for converging waves we must select the minus sign ($u_2 < u_1$, $D < u_1$).

Let us write the equation along the characteristic in the form (2.3):

$$\frac{dp}{\rho c} \pm du = F dt.$$

Instead of p, ρ and c, replacing these expressions by v and z, we readily /86 obtain

$$(h-1)\, v\, dz \pm du = F dt. \tag{11.7}$$

Due to the fact that it is written in finite differences, this relationship provides us with the equation between z and u at point 2 (Figure 27). Therefore, it can be solved concurrently with the first equation (11.6). This may be done as follows. Let us define a certain initial value of z_2, and let us calculate u_2 from (11.6) on the one hand, and from (11.7) on the other hand. Let us compare the results obtained and, changing z_2, we find that they coincide. We may employ linear interpolation here; after two "tests" are performed, the third value of z_2 is determined by interpolation over the first two, so that the difference between the two values of u_2 vanishes. As a rule, the number of

"tests" is not great, since the initial value of z_2 may be taken from the preceding wave point with sufficient accuracy - or, which is even better, it may be extrapolated to point 2. After u_2 and z_2 are computed, we may find v_2 from the second equation (11.6), after which point 2 "is calculated" in the normal manner.

The second method for solving the equations on the wave front employs Reimann invariants. In a certain sense, it is more comprehensive, but necessitates additional tables.

Let us multiply both parts of the second equation (11.6) by τ^2. Assuming that $x = \tau^{h+1}$, we may write

$$\left(\frac{c_2}{c_1}\right)^2 = x\frac{x+h}{hx+1},$$

or

$$\frac{c_2-c_1}{c_1} = \sqrt{x}\frac{\sqrt{x+h}}{\sqrt{hx+1}} - 1. \tag{11.8}$$

We may rewrite the first equation (11.6) in the following form

$$\frac{u_2-u_1}{c_1} = \pm\sqrt{\frac{h-1}{x}}\frac{x-1}{\sqrt{hx+1}}. \tag{11.9}$$

The upper sign is selected for a diverging wave, and the lower sign is selected for a converging wave. Multiplying (11.8) by h - 1 and combining it with (11.9), we have

$$\frac{A_2-A_1}{c_1} = (h-1)\left(\sqrt{x}\frac{\sqrt{x+h}}{\sqrt{hx+1}} - 1\right) \pm \sqrt{\frac{h-1}{x}}\frac{x-1}{\sqrt{hx+1}}. \tag{11.10}$$

In a similar way, we have

$$\frac{B_2-B_1}{c_1} = (h-1)\left(\sqrt{x}\frac{\sqrt{x+h}}{\sqrt{hx+1}} - 1\right) \mp \sqrt{\frac{h-1}{x}}\frac{x-1}{\sqrt{hx+1}}. \tag{11.11}$$

We can write the second and third equations (11.6) in the following form /87

$$\left.\begin{aligned} \frac{v_2}{v_1} &= x^{\frac{1}{2x}}\frac{\sqrt{x+h}}{\sqrt{hx+1}}, \\ \frac{D-u_1}{c_1} &= \pm\frac{\sqrt{hx+1}}{\sqrt{h+1}}. \end{aligned}\right\} \qquad (11.12)$$

Let us introduce the notation

$$X = \begin{cases} A \text{ for a diverging wave,} \\ B \text{ for a converging wave;} \end{cases}$$

$$Y = \begin{cases} B \text{ for a diverging wave,} \\ A \text{ for a converging wave.} \end{cases}$$

Thus, X is the Reimann invariant, which is integrated over the "overtaking" wave of the characteristic; Y is the other Reimann invariant. Relationships (11.10), (11.11), (11.12) now assume the following form

$$\left.\begin{aligned} \frac{X_2-X_1}{c_1} &= (h-1)\left(\sqrt{x}\frac{\sqrt{x+h}}{\sqrt{hx+1}}-1\right)+\sqrt{\frac{h-1}{x}}\frac{x-1}{\sqrt{hx+1}}, \\ \frac{Y_2-Y_1}{c_1} &= (h-1)\left(\sqrt{x}\frac{\sqrt{x+h}}{\sqrt{hx+1}}-1\right)-\sqrt{\frac{h-1}{x}}\frac{x-1}{\sqrt{hx+1}}, \\ \frac{v_2}{v_1} &= x^{\frac{1}{2x}}\frac{\sqrt{x+h}}{\sqrt{hx+1}}, \\ \pm\frac{D-u_1}{c_1} &= \frac{\sqrt{hx+1}}{\sqrt{h+1}}. \end{aligned}\right\} \qquad (11.13)$$

Since the quantities on the left depend only on x, they are all functionally interrelated. Therefore, we may compile a table in the following form:

$\frac{X_2-X_1}{c_1}$	$\frac{Y_2-Y_1}{c_1}$	$\frac{v_2}{v_1}$	$\pm\frac{D-u_1}{c_1}$
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .

The equations may now be solved in the following way. After the lower quantities (appearing in the table under the index 1) are determined at point 2, we may assign a certain value to v_2. This enables us to integrate the Reimann invariants over the "overtaking" characteristic, i.e., to determine X_2. We thus obtain an "entry" in the table, from which we may find a new

value of v_2. We may employ it to find the new value of X_2, and the new /88 value of v_2 from the table, et cetera. As a rule, this process is one which converges rapidly, and we finally obtain the values for X_2 and v_2. From the same table we may now find Y_2 and D, after which we may assume that point 2 has been computed.

We should point out that the left sides of relationships (11.13) contain the same parameter - the adiabatic index $\varkappa$ (because $h = \frac{\varkappa + 1}{\varkappa - 1}$). Therefore, this table is suitable for all ideal gases with the given adiabatic index - i.e., it has quite universal application.

Let us turn briefly to substances with an equation of state such as (5.15). The expression for the internal energy in this case may be written as

$$\varepsilon = \frac{p + \rho_0 c_0^2}{(\varkappa - 1)\rho}. \tag{11.14}$$

[It follows from equation (5.15) that the internal energy is related to pressure, density, and entropy by the equation

$$\varepsilon = \frac{p + \rho_0 c_0^2}{(\varkappa - 1)\rho} + f(k).$$

where f(k) is an arbitrary function of entropy. However, in all cases which are justified in physical terms, we may assume f(k) = const. Because the internal energy is determined within the accuracy of an additive constant, we may employ formula (11.14)]. If we now perform all the corresponding computations, it may be readily confirmed that equation (11.6) remains in force (we should recall that z and v are now determined by formulas (5.19) and (5.21). The relationships (11.13) remain unchanged, and consequently the table described above, which may be employed both for substances such as an ideal gas and for substances such as (5.15) (it is only important that the adiabatic indices $\varkappa$ coincide for them). Thus, the computation of shock waves for equations of state such as (5.15) barely differs from their computation for ideal gases.

In certain cases, the state of motion of a medium "unperturbed" by a shock wave (i.e., motion in the lower region) may be given, not by the grid of the characteristics, but rather analytically, in the form of formulas. The most important case here is the calculation of a shock wave propagated in a substance at rest, i.e., the case when the lower quantities have the values

$$u = 0, \quad p = \text{const}, \quad \rho = \text{const}.$$

The difficulty entailed in the interpolation of the lower data is then eliminated, and the entire computation is simplified accordingly. The second computational method may always be employed here. Strictly speaking, in /89 this case the shock wave may be regarded as an unusual boundary condition. In principle, the computational procedure is the same as that described in section

7. The only difference consists of the fact that $\frac{dr}{dt} \neq u$ along the boundary, and therefore R and v are not constant on the boundary. The method for computing these waves is quite clear, and we shall not dwell on it in detail. We would only like to point out one special case.

Let us assume that we are dealing with an ideal gas, and that its initial state (before the passage of the shock wave) is characterized by the following values of velocity, density, and pressure:

$$u_1 = 0, \ \rho_1 = \rho_0 = \text{const}, \ p_1 = 0. \tag{11.15}$$

It may be stated in formal terms that condition (11.15) corresponds to an absolute zero temperature. In practical terms, an investigation of this state is valid if the shock wave passing through the gas is so strong that the pressure before its front p_1 may be disregarded. Equations (11.6) are now unsuitable, because the parameter τ becomes infinite, and we must derive the relationships between A, B, v, et cetera, directly from formulas (9.13) again. For purposes of simplification, we shall omit the index 2 for the upper quantities.

It follows from the Hugoniot adiabatic equation (11.2), which remains in force, that

$$\rho = h\rho_0; \tag{11.16}$$

and thus, the density at the shock wave front is constant. In addition, the second equation (9.13) yields

$$u^2 = -p\left(\frac{1}{h\rho_0} - \frac{1}{\rho_0}\right) = \frac{h-1}{h\rho_0}p,$$

or

$$p = \frac{h}{h-1}\rho_0 u^2. \tag{11.17}$$

From (11.16) and (11.17), we now have

$$c^2 = \varkappa\frac{p}{\rho} = \frac{\varkappa}{h-1}u^2,$$

or, after extracting the square root, we have

$$c = \pm\sqrt{\frac{\varkappa}{h-1}}u. \tag{11.18}$$

The plus sign before the root corresponds to a diverging wave, and the minus sign corresponds to a converging wave. In addition, we have

$$A = (h-1)c + u = \pm\sqrt{\frac{\varkappa}{h-1}}(h-1)u + u = (1 \pm \sqrt{h+1})u, \tag{11.19}$$

and, similarly /90

$$B = (-1 \pm \sqrt{h+1})\,u. \tag{11.20}$$

Introducing, just as before, the quantities X and Y, we obtain

$$\begin{aligned} X &= \pm(\sqrt{h+1}+1)\,u, \\ Y &= \pm(\sqrt{h+1}-1)u \end{aligned} \tag{11.21}$$

or, finally,

$$Y = \frac{\sqrt{h+1}-1}{\sqrt{h+1}+1}X. \tag{11.22}$$

In addition, from (11.16) and (11.17) we have

$$v^{2\varkappa} = k = \frac{p}{\rho^{\varkappa}} = \frac{\frac{h}{h-1}\rho_0 u^2}{h^{\varkappa}\rho_0^{\varkappa}} = \frac{u^2}{(h-1)(h\rho_0)^{\varkappa-1}},$$

or

$$v^{\varkappa} = \frac{|u|}{\sqrt{h-1}\,(h\rho_0)^{\frac{\varkappa-1}{2}}}.$$

On the basis of (11.21), we obtain

$$v^{\varkappa} = \frac{|X|}{\sqrt{h-1}\,(\sqrt{h+1}+1)\,(h\rho_0)^{\frac{\varkappa-1}{2}}},$$

i.e.,

$$v = \frac{|X|^{\frac{1}{\varkappa}}}{[\sqrt{h-1}\,(\sqrt{h+1}+1)]^{\frac{1}{\varkappa}}(h\rho_0)^{\frac{1}{h+1}}}. \tag{11.23}$$

Finally, the first relationship (9.13) yields

$$D = \frac{h}{h-1}u$$

or, according to (11.21), we have

$$D = \pm \frac{h}{(h-1)(\sqrt{h+1}+1)} X. \tag{11.24}$$

Thus, we have reduced the expressions for the shock wave front to the form of (11.22), (11.23), (11.24): /91

$$\left.\begin{aligned} v &= \frac{|X|^{\frac{1}{\varkappa}}}{|\sqrt{h-1}\,(\sqrt{h+1}+1)|^{\frac{1}{\varkappa}}\,(h\rho_0)^{\frac{1}{h+1}}}, \\ Y &= \frac{\sqrt{h+1}-1}{\sqrt{h+1}+1} X, \\ D &= \pm \frac{h}{(h-1)(\sqrt{h+1}+1)} X. \end{aligned}\right\} \tag{11.25}$$

These formulas replace relationships (11.13). Since the second and third formulas show that Y and D are simply proportional to X, it is not necessary to tabulate Y and D (as functions of X). With respect to the first formula, the table is useful here, due to the presence of a fractional power for $|X|$. It is true that the function $v = v(|X|)$ depends on ρ_0, just as on the parameter, and therefore cannot have universal application. Therefore, we may confine ourselves to the table for determining $|X|^{1/\varkappa}$.

Up to this point we have not mentioned the computation of R for the shock wave. The reasons for this are clearly apparent - this quantity does not change on the discontinuity line, but is directly transposed from the lower region. In particular, when a shock wave passes through a medium at rest with the initial density $\rho = \rho_0$, this quantity ρ_0 must be introduced in the integration formula of the Lagrangian coordinate (1.17). Then R = r will hold on the line of the shock wave front (since this equality holds in the lower region). We shall discuss control integration of R along the wave front line at a later point.

The quantity v changes across the shock wave. Thus, the shock wave produces a new functional dependence v=v(R), which may be employed when computing the upper region. A table of the function v(R) may be compiled almost simultaneously with the computation of the shock wave. A new line is added to this table with the calculation of each succeeding point of the wave.

The method presented in this section is only applicable to fairly strong shock waves. Several difficulties are encountered when applying it to the case of weak waves. This will be discussed in Section 13, where a practical method for computing weak shock waves will be presented.

12. DECAY OF AN ARBITRARY DISCONTINUITY /92

Let us assume that the initial data for the problem contain a discontinuity. If this discontinuity belongs to one of the three types described above (initial discontinuity of an expansion wave, contact discontinuity, shock wave), the subsequent motion may be computed on the basis of the rules presented in the preceeding sections. However, there may be a discontinuity for which the quantities on both sides do not satisfy even one of the three types of relationships presented above (a so-called arbitrary discontinuity). It is known from hydrodynamics that such a discontinuity rapidly decays into three (generally speaking) discontinuities, each of which belongs to one of the types described above. In other words, an arbitrary discontinuity may always be separated (by one single method) into the sum of three discontinuities, each of which is either the initial discontinuity of the expansion wave, a contact discontinuity, or a shock wave. In accordance with this, three discontinuity

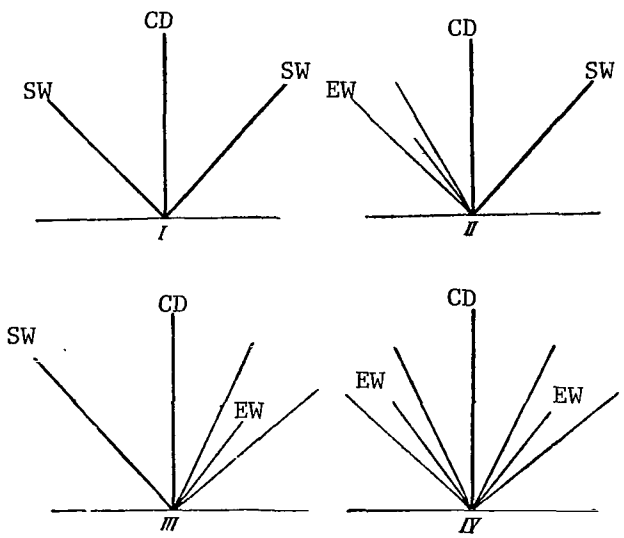


Figure 30

lines (if, for purposes of generality, we also call the region of the /93 expansion wave the "discontinuity line") will emanate from the point of the arbitrary discontinuity on the r, t-plane.

Thus, four main types of an arbitrary discontinuity decay are possible, which are shown in Figure 30. The following notation is used here: The horizontal line - initial data; CD - contact discontinuity; EW - expansion wave; SW - shock wave. There may be cases in which one of the three discontinuities drops out (its amplitude vanishes). If the initial discontinuity satisfies the conditions of one of the three main ones (CD, EW, or SW), then the "decay" may be reduced to one discontinuity.

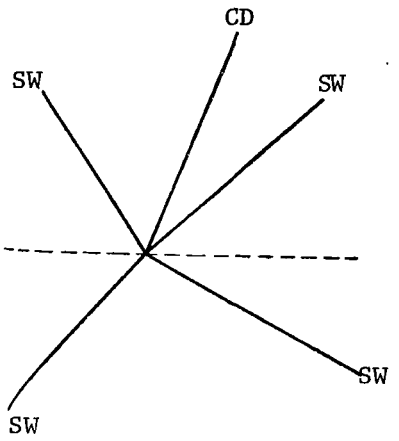


Figure 31

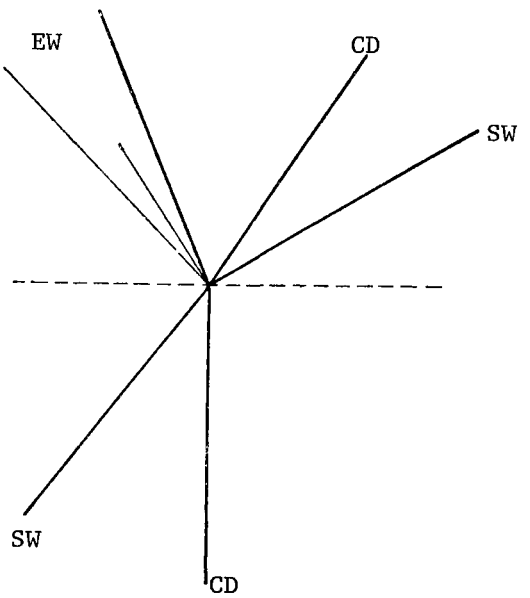


Figure 32

A type I discontinuity arises, for example, when diverging and converging shock waves come in contact (Figure 31, the dashed line designates the line which may be assumed as the line of initial data for the subsequent stage of the problem). We should point out that if two initial shock waves have identically the same amplitudes at the moment when they come in contact (i.e., if, for

example, the upper values of p coincide for them), then there is no contact discontinuity after the decay.

If the shock wave reaches the contact discontinuity, the subsequent motion belongs to one of the first three types (Figure 30). Figure 32 presents an example. It is rather difficult to formulate the conditions which are sufficient and requisite for effectuating a certain possibility in the general case*. /94 It may be stated that the case shown in Figure 31 (i.e., the case when a "reflected" expansion wave is formed) occurs, as a rule, when the shock wave changes from a denser substance (or a more "rigid") to a less dense substance (or less "rigid"). In the opposite case, a reflected shock wave is formed (Figure 30, I). The situation is the same, if one shock wave overtakes another. Case IV (Figure 30) only represents an exception when encountered in practice.

The computation of the discontinuity decay may be divided into three stages:

(1) Determination of the qualitative nature of the discontinuity (i.e., whether it belongs to one of the types indicated in Figure 30);

(2) Calculation of the initial values for all three discontinuities;

(3) Computation of the region directly adjacent to the point of the discontinuity decay.

With respect to the first point, it must be noted that the result of the discontinuity decay may be frequently predicted directly, without any additional calculations. This pertains to the "indisputable" cases which are similar to the meeting of two shock waves, which was mentioned above. Case I is also realized when two layers of a substance collide with each other. If at any moment of time two masses of substance, which are stationary with respect to each other and which are under different pressures (for example, if the thin membrane separating them is suddenly removed) come in contact, then a type II or III discontinuity decay occurs as a function of the location where there was greater pressure. On the basis of a certain amount of experience, it is possible to predict the result produced when a shock wave passes through a contact discontinuity, especially if the densities of the substance differ greatly on both sides of the contact discontinuity.

Nevertheless, cases are encountered when it is difficult to predict the nature of the discontinuity decay beforehand. Therefore, we shall describe more precise methods.

One of these methods, which was presented in the book by Courant and Friedrichs, is as follows.

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* See Courant and Friedrichs. Supersonic Flow and Shock Waves. Chpt. III, Moscow, 1950.

Let us first examine the region lying to the right of the contact discontinuity which is produced after the decay. Either a diverging shock wave may appear here, or a "diverging" expansion wave, i.e., an expansion wave through which (at the initial moment) the quantity B remains unchanged. If we write the relationship for the shock wave in the form of (11.4), replacing τ^{h+1} by $\frac{p_2}{p_1}$, after simple transformations we obtain

$$(u_2 - u_1)^2 = \frac{h-1}{\rho_1} \cdot \frac{(p_2 - p_1)^2}{(hp_2 + p_1)},$$

or, after extracting the square root (we should recall that the wave under consideration is a diverging wave), we have /95

$$u_2 = u_1 + \sqrt{\frac{h-1}{\rho_1}} \cdot \frac{p_2 - p_1}{\sqrt{hp_2 + p_1}}. \qquad (12.1)$$

We have the following through the expansion wave (retaining the previous notation for regions 1 and 2)

$$B_2 = B_1, \; v_2 = v_1.$$

Taking into account the relationships

$$B = (h-1)c - u,$$
$$c = vz,$$
$$z = \sqrt{\varkappa}\, p^{\frac{1}{h+1}},$$

we can write

$$(h-1)\sqrt{\varkappa}\, v_1 \cdot p_2^{\frac{1}{h+1}} - u_2 = (h-1)c_1 - u_1,$$

or

$$u_2 = u_1 + (h-1)\left(\sqrt{\varkappa}\, v_1 p_2^{\frac{1}{h+1}} - c_1\right). \qquad (12.2)$$

It must be assumed that the quantities with the index 1 are known - these will be the "right" quantities at the discontinuity point.

Since $p_2 > p_1$ in the case of the shock wave, $p_2 < p_1$ in the case of the expansion wave, formulas (12.1) and (12.2) determine the functional dependence $u_2 = \phi(p_2)$ for different values of the argument p_2. We may compile a curve of u_2, p_2 (Figure 33) which combines both these formulas.

Point 1 corresponds to values of u_1, p_1, which are general for both cases. The upper portion of the curve corresponds to formula (12.1), and the lower portion corresponds to (12.2). The curve as a whole comprises a collection of all

the possible states which may arise from state I after the passage of the shock wave or the expansion wave.

The situation is absolutely the same for the region to the left of the contact discontinuity for the converging shock wave, and the expansion wave, through which $A_2 = A_1$. Instead of formulas (12.1) and (12.2), we now obtain

$$\left.\begin{aligned} u_2 &= u_1 - \sqrt{\frac{h-1}{\rho_1}}\,\frac{p_2 - p_1}{\sqrt{hp_2 + p_1}}, \\ u_2 &= u_1 - (h-1)\left(\sqrt{\varkappa}\, v_1 p_2^{\frac{1}{h+1}} - c_1\right). \end{aligned}\right\} \qquad (12.3)$$

Just as previously, the indices 1 and 2 designate the lower and upper quantities, but to the left of the contact discontinuity this time. We may compile a curve according to formulas (12.3) (Figure 34). /96

The quantities p and u are continuous at the contact discontinuity. This means that if we plot both curves (Figure 33 and 34) on a general graph (Figure 35), the point at which they intersect provides us with the values of u_2, p_2 at the contact discontinuity. In this way, the nature of the discontinuity decay may be completely determined. If the curves are arranged as is shown in Figure 35, a type I discontinuity decay occurs, because the pressure

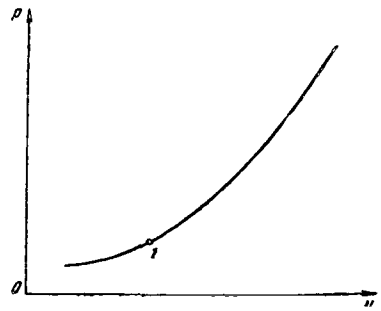

Figure 33

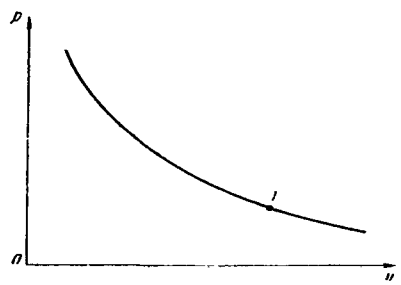

Figure 34

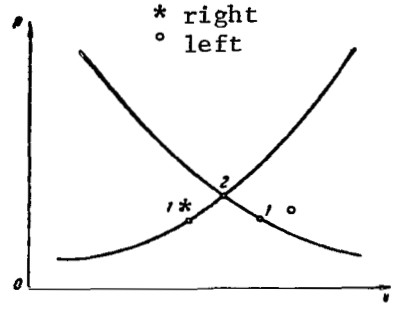

Figure 35

p on the contact discontinuity is greater than both initial values. When the curves are arranged as is shown in Figure 36, a type III occurs, since p on the contact discontinuity is less than the initial p to the right, but greater than the initial p to the left.

Although this method has no drawbacks theoretically, it is rather cumbersome. The curves p(u) depend on the initial values on both sides of the initial discontinuity, and they must be calculated and compiled anew in each specific case. Therefore, the method presented is more suitable for theoretical discussions as an instrument of proof, and is not a practical, work- /97
ing method for investigating a specific discontinuity. For this purpose, a trial calculation of the discontinuity is much more suitable. Since this method is very closely related to the method for computing the discontinuity in general, we shall now turn to this problem.

Let us assume that we are dealing with one of four types of discontinuity decay; this choice may be hypothetical. The vicinity of the discontinuity is

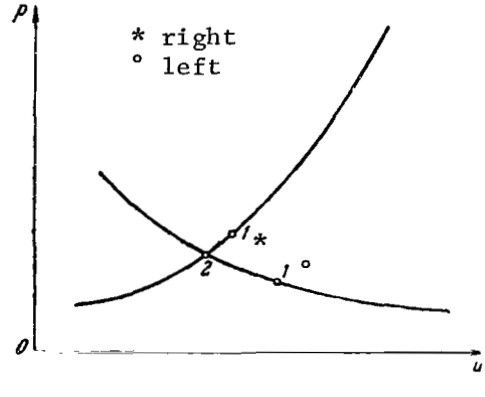

Figure 36

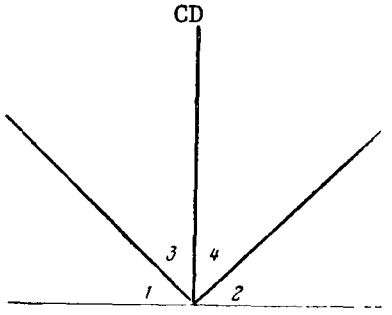

Figure 37

divided into four regions, which are designated by the indices 1, 2, 3, 4 in Figure 37. For purposes of simplicity and generality, we shall express the "lateral" discontinuities by one line, although they may be expansion waves. In the latter case, we may still exclude the region of the expansion wave from the investigation.

We know the quantities in regions 1 and 2. We must compute the quantities in regions 3 and 4 (we are speaking of the limiting values of these quantities at the discontinuity point, and not of the functions of r and t in these regions). There are six of these quantities which we are seeking - for example, u_3, p_3, ρ_3, u_4, p_4, ρ_4. It is necessary to have six equations in order to determine them. The relationships for all three "outgoing" discontinuity lines - two relationships on each line - represent these equations. Thus, the problem may be reduced to solving a system of six equations with six unknowns.

If we are dealing with equations of state of the ideal gas type, we may immediately reduce the problem to solving one single equation. In order to do this, we must employ the relationships (12.1), (12.2) and (12.3) which we obtained above. Let us write them again, changing the indices in accordance with Figure 36 and omitting the indices for the quantities $p_3 = p_4$ and $u_3 = u_4$: the right shock wave /98

$$u = u_2 + \sqrt{\frac{h-1}{\rho_2}}\,\frac{p-p_2}{\sqrt{hp+p_2}}; \tag{12.4}$$

the right expansion wave

$$u = u_2 + (h-1)\left(\sqrt{\varkappa}v_2 p^{\frac{1}{h+1}} - c_2\right); \tag{12.5}$$

the left shock wave

$$u = u_1 - \sqrt{\frac{h-1}{\rho_1}} \frac{p - p_1}{\sqrt{hp + p_1}}; \qquad (12.6)$$

the left expansion wave

$$u = u_1 - (h-1)(\sqrt{\varkappa}\, v_1 p^{\frac{1}{h+1}} - c_1). \qquad (12.7)$$

We must keep the fact in mind that $\varkappa$ and h may be different to the right and to the left of the contact discontinuity.

In accordance with the type of discontinuity decay which was selected, we must now select the appropriate pair from formulas (12.4) - (12.7) and equate the right parts. We obtain one equation with respect to p. Solving it according to the general rules for the numerical solution of equations (for example, by the method of inverse linear interpolation), we may determine p at the contact discontinuity.

We must now determine whether the result obtained corresponds to the type of discontinuity decay with which we started. The following inequalities must be fulfilled for the four types shown in Figure 30:

$$\text{I. } p > p_1,\ p > p_2; \quad \text{II. } p_1 > p > p_2;$$
$$\text{III. } p_1 < p < p_2; \quad \text{IV. } p < p_1,\ p < p_2.$$

If the value we obtain for p does not satisfy the corresponding inequality, this means that our initial hypothesis regarding the type of discontinuity decay has been refuted. We must repeat the calculation, selecting a new type corresponding to those inequalities which were satisfied by the value of p obtained the first time. In the overwhelming majority of cases, the second result is valid. We may now find u on the contact discontinuity from the appropriate formulas (12.4) - (12.7) (it is almost always calculated during the process of solving the equation for p). We may then find the values of v_3 and v_4. They are simply transferred from regions 1 or 2 through the expansion wave, and formula (11.3) may be employed for the case of the shock wave. We can find z_3 and z_4 and the remaining quantities on both sides of the contact discontinuity.

This is the manner in which the discontinuity decay is calculated for ideal gases. The situation is not much more complex for the equation of state such as (5.15). The relationship for the shock wave may be reduced to the following form

$$u_2 = u_1 \pm \frac{c_1}{c_2} \sqrt{\frac{h+1}{\rho_0 z_1^{h+1}}} \cdot \frac{p_2 - p_1}{\sqrt{\varkappa(hp_2 + p_1) + (h+1)\rho_0 c_0^2}} \qquad (12.8)$$

and it may be reduced to the following through the expansion wave /99

$$u_2 = u_1 \pm (h-1)\left[v_1\left(\frac{\varkappa}{\rho_0 c_0^2}\,p_2 + 1\right)^{\frac{1}{h+1}} - c_1\right]. \qquad (12.9)$$

In formulas (12.8) and (12.9) the signs are selected, depending on which side of the contact discontinuity the shock wave or the expansion wave are located: the plus sign is selected for the right side, and the minus sign is selected for the left side. The index 2 designates u and p on the contact discontinuity, and the index 1 indicates the corresponding (right or left) lower quantities. The rest of the calculation proceeds in the manner described above.

For an equation of state such as (5.22), the discontinuity decay is complicated by the fact that the Riemann invariant $\int\frac{dp}{\rho c} \pm u$ is not expressed in the form of the elementary functions of u and c (see Section 5). Therefore, the system of equations determining the discontinuity decay include the following equation in the case of an expansion wave, at least on one side of the contact discontinuity:

$$\int_{p_1}^{p_2}\frac{dp}{\rho c} \pm u_2 = \pm u_1.$$

Here, the upper limit p_2 is unknown, and the integral is found by numerical methods (see Section 5). The remaining equations of the system may be written with no difficulty similarly to the equation of state for an ideal gas.

The problem of the discontinuity decay, which we are now investigating, is closely related to the problem of reflection of shock waves from the boundaries investigated in Section 7. A reflected wave arises after the shock wave has reached such a boundary. Depending on the boundary conditions, it may be both a shock wave (Figure 38) and an expansion wave (Figure 39). Let us briefly discuss the two types of boundary conditions described in Section 7.

If a condition such as $u = u(t)$ is given on the boundary, where the function u(t) is continuous, then the situation shown in Figure 38 occurs, i.e., the reflected wave will be a shock wave. In actuality, since u on the boundary was given beforehand, we have $u_1 = u_3$ (Figure 38). In addition, we will have $u_2 > u_1$ on the shock wave "approaching" the boundary; therefore, $u_2 > u_3$, and this latter inequality is characteristic, as we know, for a shock wave.

Formulas (11.6) may be employed to compute the initial point of the reflected wave (the indices 1 and 2 are replaced by the indices 2 and 3, according to Figure 37). We know the left part of the first formula, and we can determine τ^{h+1} (in order to do this, we must solve a quadratic equation), as well as z_3. The second formula gives us v_3, and we may assume that the desired point has been computed.

In the second case, the pressure is given on the boundary in the /100

form of the continuous function $p(t)$. In this case, $p_3 = p_1$, and it may be readily ascertained that the reflected wave is an expansion wave. Since $p_2 > p_1$, then $p_3 < p_2$. Thus, the case shown in Figure 35 has been realized. In order to compute the expansion wave which is produced, it is sufficient to determine the jump in A, i.e., the value of A_3. The following procedure may be followed. Knowing p_3, we may immediately obtain z_3, and since $v_2 = v_3$, $c_2 = v_3 z_3$. However, we now have $B_3 = B_2$, and we may thus find A_3:

$$A_3 = 2(h-1)c_3 - B_3.$$

The method described in Section 8 may now be employed to calculate the expansion wave, after which the normal procedure may be followed in computing the boundary.

Let us calculate the region directly adjacent to the discontinuity point. It can be readily seen that the methods presented in the preceding sections are inapplicable here, since they assume that a rather long characteristic may be drawn from the contact discontinuity point which has already been computed or the shock wave. It is impossible to draw any characteristics from the initial point of the discontinuity decay. Therefore, so long as the lines of the new discontinuities do not "diverge" sufficiently, special methods must be employed.

For purposes of definition, let us assume that our discontinuity decayed according to type III (Figure 30) - i.e., that a shock wave is formed to the left, and an expansion wave is formed to the right. We may compute the region of the expansion wave independently, employing the method presented in /101 Section 8. From this point on, we need only deal with its upper characteristic, which we may assume to be calculated as far as we require.

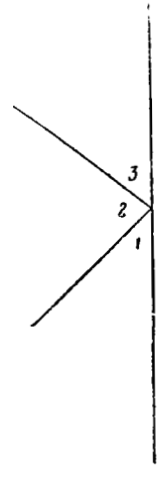

Figure 38

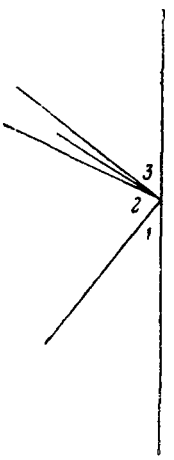

Figure 39

Let us draw the line dr = u dt from the initial point 0 (Figure 40); it depicts the line of the contact discontinuity. Let us select point 1 on this line so close to point 0 that every quantity may be transposed from point 0 to point 1 without a great amount of error. Drawing the β-characteristic to the left of point 1, we may compute point 2 of the shock wave front in the normal procedure.

Let us place point 3 on the upper characteristic of the expansion wave, and let us draw the β-characteristic from it until it reaches the contact discontinuity (point 4). Let us draw the α-characteristic from point 2; this characteristic intersects the contact discontinuity line at point 5. By integration over these segments of the characteristics, we may compute A_5 and B_4, and then - performing linear interpolation with respect to points 0 and 5 - we may calculate the left A at point 4 (we assume that points 4 and 5 are located as is shown in Figure 40, otherwise we must interpolate the right B at point 5). We may compute point 4 in the normal way.

After this, let us define point 1 more accurately, interpolating every quantity at it over the points 0 and 4. Then let us repeat the entire computation in the same order, only adding one internal point 6, calculated after the recalculation of point 2. If it is found when the calculation is repeated that point 4 has changed considerably, we repeat the calculation once more until all of our six points are "determined".

We would like to say a few words regarding the calculation of the lower quantities at point 2. If the "lower" characteristic passes nearby, we must then select point 1 so that point 2 falls on this characteristic. If /102 there is no such suitable characteristic, we must "create" it, performing suitable "subdividing" in the lower data.

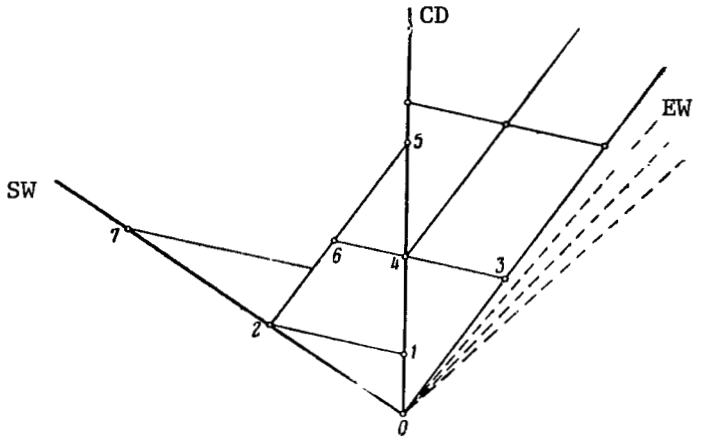

Figure 40

After points1 - 6 are computed, the subsequent computation entails no particular difficulties. On the basis of the 2 - 6 - 5 characteristic, we may compute a new point 7 of the shock wave. Bringing the α-characteristic to the right of point 4, we may move the β-characteristic to the right, closer to the contact discontinuity line, and we may then complete the calculation of point 5. Let us then bring the α-characteristic from point 7, and continue the calculation of the shock wave, et cetera.

If there is no expansion wave on the right, but only a shock wave, then we would bring the characteristic also on the right from point 1, and would compute the point of the right shock wave, which would play the role of point 3.

At the very beginning of the computation, we must perform linear interpolation along the contact discontinuity line. Consequently, the first steps (0 - 4, 0 - 5) must be so small that the error entailed in linear interpolation is insignificant. The choice of point 1 is thus determined.

Along with the method presented above, there is another possible method for computing the vicinity of the discontinuity decay point, which is based on expansion of the solution in power series in the vicinity of this point.

Let us first present certain preliminary considerations. Let us assume that at a certain point on the r, t-plane we know the derivatives of u, ρ, p in any direction $\frac{dr}{dt} = m$ which does not coincide with the characteristic direction ($m \neq \alpha$, $m \neq \beta$, $m \neq u$). We may then determine the derivatives of these quantities in any other direction. In actuality, let us designate the derivatives

in the direction m by Δu, $\Delta\rho$, Δp:

$$\Delta u = \frac{\partial u}{\partial t} + m\frac{\partial u}{\partial r},$$
$$\Delta\rho = \frac{\partial \rho}{\partial t} + m\frac{\partial \rho}{\partial r},$$
$$\Delta p = \frac{\partial p}{\partial t} + m\frac{\partial p}{\partial r}.$$

We thus have

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u - m\frac{\partial u}{\partial r},\\ \frac{\partial \rho}{\partial t} &= \Delta\rho - m\frac{\partial \rho}{\partial r},\\ \frac{\partial p}{\partial t} &= \Delta p - m\frac{\partial p}{\partial r}.\end{aligned}\right\} \qquad (12.10)$$

Let us write the differential equations of motion in the form (2.2): /103

$$\left.\begin{aligned}\frac{1}{\rho c}\left(\frac{\partial p}{\partial t} + \alpha\frac{\partial p}{\partial r}\right) + \left(\frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial r}\right) &= -\nu\frac{uc}{r},\\ \frac{1}{\rho c}\left(\frac{\partial p}{\partial t} + \beta\frac{\partial p}{\partial r}\right) - \left(\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial r}\right) &= -\nu\frac{uc}{r}\end{aligned}\right\} \qquad (12.11)$$

and let us substitute their expressions from (12.10), instead of $\frac{\partial p}{\partial t}$, $\frac{\partial u}{\partial t}$. After elementary transformations, we obtain

$$\left.\begin{aligned}\frac{(\alpha - m)}{\rho c}\frac{\partial p}{\partial r} + (\alpha - m)\frac{\partial u}{\partial r} &= -\left(\nu\frac{uc}{r} + \frac{\Delta p}{\rho c} + \Delta u\right),\\ \frac{(\beta - m)}{\rho c}\frac{\partial p}{\partial r} - (\beta - m)\frac{\partial u}{\partial r} &= -\left(\nu\frac{uc}{r} + \frac{\Delta p}{\rho c} - \Delta u\right).\end{aligned}\right\} \qquad (12.12)$$

We may regard this system as a system of linear equations with respect to $\frac{\partial p}{\partial r}$, $\frac{\partial u}{\partial r}$. Its determinant is

$$\frac{-2(\alpha - m)(\beta - m)}{\rho c},$$

and, by definition, differs from zero.

Solving it, we obtain

$$\frac{\partial u}{\partial r} = \frac{c\left(\nu\frac{uc}{r} + \frac{\Delta p}{\rho c}\right) - (u-m)\,\Delta u}{(\alpha - m)(\beta - m)},$$

$$\frac{\partial p}{\partial r} = \rho c\,\frac{c\Delta u - (u-m)\left(\nu\frac{uc}{r} + \frac{\Delta p}{\rho c}\right)}{(\alpha - m)(\beta - m)}. \qquad (12.13)$$

In addition, the following equation holds [see the derivation of equation (2.2)]:

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} = c^2\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r}\right).$$

We find the following from this equation in a similar manner

$$\frac{\partial \rho}{\partial r} = \frac{1}{c^2}\frac{\partial p}{\partial r} + \frac{\Delta p - c^2\Delta\rho}{c^2(u-m)}. \qquad (12.14)$$

Relationships (12.13), (12.14) enable us to find all of the partial derivatives with respect to r, and relationships (12.10) enable us to find the partial derivatives with respect to t. Knowing all the partial derivatives of u, ρ, p, we may readily determine their derivatives in any direction.

In particular, introducing the standard notation $\frac{\partial}{\partial t} + u\frac{\partial}{\partial r} = \frac{d}{dt}$, we /104 obtain the following for the direction $\frac{dr}{dt} = u$

$$\left.\begin{aligned}\frac{du}{dt} &= \left[1 - \frac{(u-m)^2}{(\alpha-m)(\beta-m)}\right]\Delta u + \frac{1}{\rho}\,\frac{u-m}{(\alpha-m)(\beta-m)}\,\Delta p + \\ &\quad + \nu\,\frac{u-m}{(\alpha-m)(\beta-m)}\,\frac{uc^2}{r},\\ \frac{dp}{dt} &= \left[1 - \frac{(u-m)^2}{(\alpha-m)(\beta-m)}\right]\Delta p + \rho c^2\,\frac{u-m}{(\alpha-m)(\beta-m)}\,\Delta u - \\ &\quad - \nu\,\frac{(u-m)^2}{(\alpha-m)(\beta-m)}\,\frac{\rho u c^2}{r}.\end{aligned}\right\} \qquad (12.15)$$

Let us now investigate the case of characteristic direction. For example, let $m = \alpha$. Then, setting $\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial r} = \Delta_\alpha$, from the second equation (12.12) we obtain

$$2c\left(\frac{1}{\rho c}\frac{\partial p}{\partial r} - \frac{\partial u}{\partial r}\right) = \nu\frac{uc}{r} + \frac{\Delta_\alpha p}{\rho c} - \Delta_\alpha u,$$

and from (12.10)

$$\frac{du}{dt} = \Delta_\alpha u - c\frac{\partial u}{\partial r},$$

$$\frac{dp}{dt} = \Delta_\alpha p - c\frac{\partial p}{\partial r}.$$

Thus, excluding $\frac{\partial u}{\partial r}$ and $\frac{\partial p}{\partial r}$, we arrive at the relationship

$$\frac{1}{\rho c}\frac{\partial p}{\partial t} - \frac{du}{dt} = -\frac{1}{2}\left(\nu\frac{uc}{r} - \frac{\Delta_\alpha p}{\rho c} + \Delta_\alpha u\right). \tag{12.16}$$

In addition, as always along the α-characteristic we have

$$\frac{\Delta_\alpha p}{\rho c} + \Delta_\alpha u = -\nu\frac{uc}{r}. \tag{12.17}$$

We must now obtain the relationship between the derivatives along the shock wave front. We indicated above that a relationship such as (12.1) holds for the shock wave; this relationship contains only u, p on both sides of the discontinuity. Differentiating it, we arrive at the desired equation. For example, if our wave moves in an ideal gas, then - differentiating (12.1) and designating the derivatives along the wave front by the symbol Δ - we obtain

$$\frac{\Delta u_2 - \Delta u_1}{u_2 - u_1} = \frac{[hp_2 + (h+2)p_1]\Delta p_2 - [(2h+1)p_2 + p_1]\Delta p_1}{2(p_2 - p_1)(hp_2 + p_1)} - \frac{\Delta\rho_1}{2\rho_1}. \tag{12.18}$$

It may be readily seen that this equation is equally valid both for converging, and for diverging, waves. /105

Let us now turn to the discontinuity decay shown in Figure 40. We shall assume that we know the derivatives of the lower quantities u and p along the front of the shock wave emanating to the left, as well as the derivativaes of the same quantities along the upper characteristic of the expansion wave. The formulas which we obtained above then enable us to determine the derivatives of these quantities along the wave front from above and along the contact discontinuity.

Relationship (12.16) is valid to the right of the contact discontinuity, and equations (12.15) are valid to the left of it. The symbol Δ may be employed to designate the derivatives along the wave front (from above). The quantities $\frac{du}{dt}$, $\frac{dp}{dt}$ coincide in both formulas. Substituting their expressions from (12.15) into (12.16), we obtain an equation (linear) which relates the derivatives Δu, Δp. The second equation for these derivatives is given by formula (12.18). Solving these equations, we may find Δu and Δp, and then from (12.15) we find $\frac{du}{dt}$ and $\frac{dp}{dt}$. After this, no difficulty is encountered in

determining the derivatives $\Delta\rho$ and $\frac{d\rho}{dt}$.

We may now write the following along the contact discontinuity

$$u = u_0 + \frac{du}{dt}(t - t_0) + \dots,$$

$$p = p_0 + \frac{dp}{dt}(t - t_0) + \dots,$$

$$\rho = \rho_0 + \frac{d\rho}{dt}(t - t_0) + \dots,$$

where the index 0 designates the quantities at point 0. Similarly, along the wave front we have

$$u = u_0 + \Delta u\,(t - t_0) + \dots.$$
$$p = p_0 + \Delta p\,(t - t_0) + \dots.$$
$$\rho = \rho_0 + \Delta\rho\,(t - t_0) + \dots.$$

The equation for the contact discontinuity line may be written as follows

$$r = r_0 + u_0(t - t_0) + \frac{1}{2}\frac{du}{dt}(t - t_0)^2 + \dots,$$

and the equation for the shock wave front may be written as

$$r = r_0 + D_0(t - t_0) +$$
$$+ \frac{1}{2}\left[\Delta u_1 + (D_0 - u_1)\left(\frac{\Delta p_2 - \Delta p_1}{p_2 - p_1} - \frac{\Delta u_2 - \Delta u_1}{u_2 - u_1}\right) - \frac{\Delta\rho_1}{\rho_1}\right](t - t_0)^2 + .$$

(we shall leave the derivation of this formula to the reader).

By employing this line of reasoning, we may find both the point on /106 the contact discontinuity, and the point on the shock wave (these points must be quite close to point 0), after which the surroundings of the discontinuity decay may be readily calculated. When necessary, all partial derivatives of u, p, ρ on both sides of the contact discontinuity may be readily determined, and they may be employed to compute the inner points which do not lie on the discontinuities.

The derivatives of u, p, ρ along the upper characteristic of the expansion wave and along the shock wave front (below) must be determined by numerical differentiation. Along the upper characteristic of the expansion wave, it is sufficient to differentiate u and ρ, for example; the derivative $\Delta^{\alpha}\rho$ may be found from relationship (12.17). In order to find the derivatives along the lower shock wave front, it is possible (provided that it is convenient) to first differentiate u, p, ρ in any direction (which does not coincide with the characteristic direction), and then to calculate the derivatives in the requisite direction by means of formulas (12.10), (12.13), (12.14).

It should be noted that both methods for computing the surroundings of the discontinuity decay point entail approximately the same degree of accuracy, because both methods assume that the functions change linearly within the surroundings of the function. The first method is most frequently preferred in practice, since it is similar to the system for performing computations in smooth regions and also utilizes the data which are already available to the best extent (numerical differentiation is not required).

13. SPECIAL CASES OF THE FORMATION OF SHOCK WAVES /107

It is well known that shock waves may arise not only from discontinuous initial data (and boundary conditions), but also in regions where the motion is not accompanied by any discontinuities at the beginning. Every such case of "spontaneous" formation of shock waves is always related to the phenomenon of the intersection of similar characteristics. The centered compression wave which was discussed in Section 8 (Figure 14, on the left) may serve as an example of this intersection. A shock wave is formed at the point r_0, t_0.

However, such a case - when all the converging characteristics intersect at one single point - is a special case, or - more precisely - a limiting case. It most frequently happens that an entire region appears which is covered twofold by the characteristics of one set (Figure 41). An envelope appears for these characteristics, having the cuspidal point A. A new shock wave begins from this point. The cuspidal point of the envelope is not a discontinuity point: it is distinguished by the fact that the derivatives of u, p, ρ, et cetera, at this point are infinite. Therefore, the amplitude of the shock wave arising at this point equals zero, and only a "real" discontinuity arises.

In practice, it is usually impossible to make a precise determination of the location of the envelope cuspidal point or to initiate the shock wave from zero amplitude. The shock wave "leaves" the point at which the characteristics of one set, which directly participate in the computation (Figure 42), first intersect. By determining the coordinates of the intersection point, we may find two values of A over the two α-characteristics arriving at this point (if they intersected). Thus, a discontinuity appears at this point.

Strictly speaking, this discontinuity must be regarded as an arbitrary discontinuity. When it decays, a contact discontinuity and, for example, an expansion wave are also formed, in addition to the shock wave (shown by the dashed line in Figure 42). However, if the intensity of our initial discontinuity is low, then these "side" effects may be disregarded.

In order to demonstrate this, let us turn to formulas (11.13). The value $x = 1$ corresponds to a wave having zero amplitude. Let us set

$$x = 1 + \varepsilon,$$

so that /108

$$\varepsilon = \frac{p_2 - p_1}{p_1}.$$

Let us expand the first parts of formulas (11.13) in power series of ε. We obtain

$$\frac{X_2 - X_1}{c_1} = \frac{2}{\varkappa}\varepsilon - \frac{h}{\varkappa(h+1)}\varepsilon^3 + \frac{7h^2}{8\varkappa(h+1)^2}\varepsilon^3 + \dots$$

$$\frac{Y_2 - Y_1}{c_1} = \frac{h^2}{8\varkappa(h+1)^2}\varepsilon^3 + \dots$$

$$\frac{v_2}{v_1} = 1 - \frac{h^2 - h - 1}{2\varkappa(h+1)^2}\varepsilon^3 + \dots$$

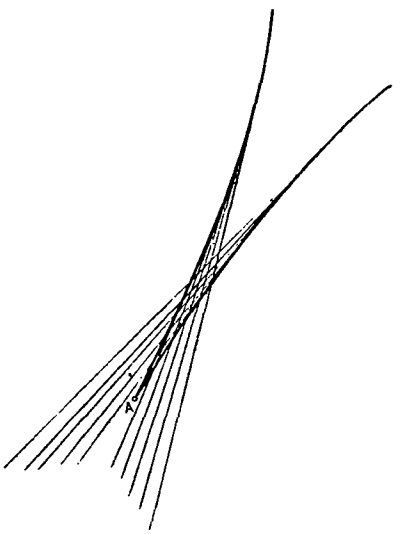

Figure 41

Thus, if we disregard terms on the order of ε^3, we shall have

$$\left.\begin{aligned} Y_2 &= Y_1, \\ v_2 &= v_1. \end{aligned}\right\} \qquad (13.1)$$

These will be the relationships on the front of the shock wave produced, if it is fairly weak (at the first point). It thus follows that the discontinuity decay may be disregarded. Relationships (13.1) are clearly fulfilled at the first point where the characteristics intersect. Therefore, it is sufficient to bring a (weak) shock wave from this point without any contact discontinuities or other effects accompanying the discontinuity decay.

We would like to point out that in actuality there is no discon- /109
tinuity decay, because relationships (13.1) are fulfilled at the apex of the envelope.

No particular difficulties are encountered in computing a shock wave which

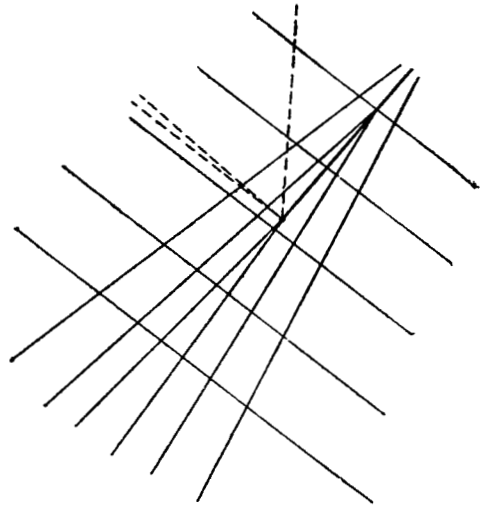

Figure 42

is thus formed. As a rule, its amplitude increases quite rapidly at the first points, until it reaches a certain "normal" level, after which it changes comparatively slowly.

Cases are possible in which the shock wave, being weak when produced, remains weak from that point on, so that relationships (13.1) continue to be fulfilled within the limits of the assumed accuracy. On the other hand, a "normal," strong shock wave can later become weak; there are possible cases in which weak shock waves are produced during a discontinuity decay. In all such cases, we must employ formulas (13.1) which are incomparably simpler than the customary relationships for the wave front. We must add that the expression for the wave velocity D for weak waves is simplified considerably. It can be readily found from formulas (11.8), (11.9) and (11.12) that

$$\frac{1}{c_1}\left[D - \frac{(u_2 \pm c_2) + (u_1 \pm c_1)}{2}\right] = \pm\left[\frac{\sqrt{hx+1}}{\sqrt{h+1}} + \right.$$

$$\left. + \frac{\sqrt{x}}{2}\,\frac{\sqrt{x+h}}{\sqrt{hx+1}} + \frac{1}{2}\sqrt{\frac{h-1}{x}}\,\frac{x-1}{\sqrt{hx+1}} + \frac{1}{2}\right].$$

Expanding the right part in powers of $\varepsilon = x - 1$, we obtain

$$\frac{1}{c_1}\left[D - \frac{(u_2 \pm c_2) + (u_1 \pm c_1)}{2}\right] = \frac{h^2}{8(h+1)^2}\varepsilon^2 + \ldots$$

Thus, within an accuracy of terms on the order of ε^2, we have

$$D = \begin{cases} \frac{\alpha_2 + \alpha_1}{2} & \text{for diverging wave} \\ \frac{\beta_2 + \beta_1}{2} & \text{for converging wave} \end{cases} \tag{13.2}$$

It is true that the accuracy of (13.2) is one order of magnitude less than that of (13.1), but it may be successfully employed in several cases.

The computation of weak shock waves has several distinguishing features. The quantity D differs very little from α (or, accordingly, β). Therefore, the "overtaking" characteristics are almost parallel to the discontinuity line, intersecting it at a very narrow angle. This fact makes it almost impossible to employ the second method for computing shock waves (Section 11, Figure 29): the coordinates of point 2 are determined with a great amount of error. The first method must always be employed when computing weak shock waves (Figure 28). We must point out that point 3 is frequently very close to point 1; therefore, one may employ linear interpolation in order to find quantities at this point.

It must be noted that it is only necessary to compute the weak /110
shock wave if, for some reason or other, it is necessary to trace all the details of its motion as accurately as possible. In general, weak waves may be disregarded. In actuality, according to formulas (13.1) the corresponding Reimann invariant and entropy do not undergo a discontinuity along the "intersecting" characteristics, and therefore they may be integrated through the discontinuity line. Thus, the necessity of an accurate determination of the wave front position is eliminated, and the calculation may proceed as though there were no discontinuity. It is true that the "overtaking" characteristics may intersect from time to time, and they must be discarded, but this fact entails no special complications.

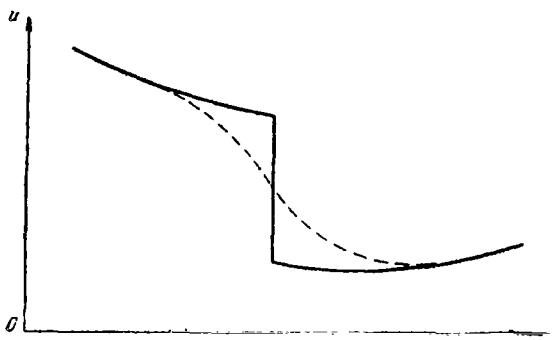

Figure 43

The phenomenon of shock wave formation with continuous initial data, which we investigated, plays an important role in the problems related to the

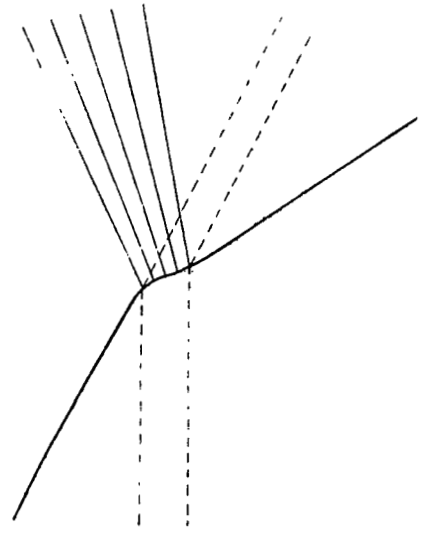

Figure 44

"smoothing out" of the discontinuities. When solving a certain problem up to a specific moment in time t, if we attempt to eliminate the shock waves by replacing the discontinuous initial data (for the subsequent stage) by the continuous data [see Figure 43; the graph of u(r) is given as an example], as the calculation proceeds all of these discontinuities are rapidly restored due to the intersection of the characteristics. Thus, it is useless to "smooth out" the strong shock waves.

This pertains only to the method of characteristics investigated here. In recent years, difference methods for numerical integration of equations of hydrodynamics have been widely discussed, beginning with the work by Neumann and Richtmeyer*. These methods are based on "smoothing out" the discontinuities. The difference equations are chosen so that the integral laws of conservation (1.19) are always fulfilled (within a certain degree of accuracy)/111 independently of the nature of the functions to be computed. The solution is everywhere continuous; instead of discontinuities, there are regions of rapid change in the quantities - the discontinuities "are smoothed out".

In certain respects, these methods are inferior to the method of characteristics. They do not enable us to follow in detail all of the discontinuities - both strong and weak - or to derive a comprehensive and detailed picture of the motion, the exact boundaries of the regions of influence, et cetera. On the other hand, they have several definite advantages, the most important of

* Neumann, J. and Richtmeyer. J. Appl. Phys. 21, 232-237, 1950. See also Godunov, S. K. Uspekhi Matem. Nauk, 12, No. 1 (73), 176-177, 1957.

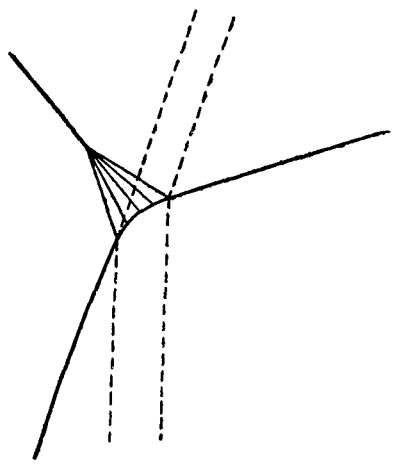

Figure 45

which is logical simplicity. The discontinuities do not differ in formal terms from the "smooth" regions, and may be computed according to the formulas which apply to them.This fact is most important when performing calculations on electronic computers.

It is fully possible that cases may arise in which one part of the problem is calculated by one method, and the other part is calculated by another method. Let us assume, for example, that the beginning of motion is computed according to difference formulas based on "smoothing out" of the discontinuities. Let us assume that the computations are performed up to a certain moment in time $t = t_0$. After this, the computation is continued by the method of characteristics for certain reasons. For this new stage, the results derived from the preceding calculations - which provide the values of the functions of u, p, ρ, etc., to be computed in the case of $t = t_0$ - will serve as the initial data.

It follows from the statements presented above that if such initial data contain smothed shock waves, then, during the subsequent computational process, these shock waves will be produced very rapidly again as "real" strong discontinuities. Therefore, before the computation is initiated using the /112 method of characteristics, it is recommended that the initial data be studied carefully, in order to discover beforehand all of the "hidden" shock waves and to convert them in the initial data into strong discontinuities in order to eliminate the necessity of dealing with the intersection of characteristics at a later point.

This unusual "stability" of the shock waves does not extend to the contact discontinuities. If such a decay is smoothed out, it is not "spontaneously"

restored. It does not follow from this, however, that - for example, when a shock wave passes through such a "hidden" contact discontinuity - the phenomena accompanying the shock wave passage through a "real" discontinuity will be completely absent. If, for example, we try to smooth out the contact discontinuity shown in Figure 32, when the shock wave passes through it the picture shown in Figure 44 is obtained. The region of the "diffused" contact discontinuity is shown by the dashed line. As we may see, the difference lies entirely in the fact that the expansion wave passing to the left is not a centered wave. If the nature of the discontinuity decay were different - if the shock wave passed to the left - we would obtain the picture shown in Figure 45. "The left" shock wave arises due to the intersection of the characteristics. As can be seen, no important simplifications are introduced when the contact discontinuity is smoothed out. It must be added that if the region of the smoothed-out contact discontinuity is as strong as desired, it will contain large gradients of entropy, density, and other quantities. This causes the error to increase and leads to the necessity of decreasing the step in this region. Therefore, we cannot recommend that the strong contact discontinuities be smoothed out. The weak, small, and unimportant discontinuities may be, and frequently must be, disregarded.

14. SEPARATION /113

As we have already pointed out, the possible formation of zero and negative pressures designating the phenomenon of tension is a characteristic feature of substances with an equation of state such as (5.15). In all such cases we must consider the possibility of destruction of the substance.

The problem of how much tension a certain substance may sustain under given conditions is quite complex, and we shall not investigate it. We shall select only one, rather idealized scheme for the phenomenon under consideration. We shall assume that our substances cannot sustain negative pressures, in general. This assumption may be regarded as valid when we are dealing with motion accompanied by such large positive and negative pressures that the tensions, which our substance can sustain, can be disregarded.

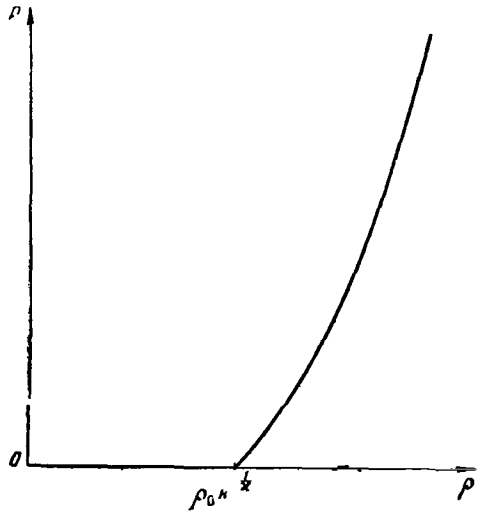

Figure 46

From a purely thermodynamic point of view, it may be assumed that we are investigating a substance with the following equation of state:

$$p = \begin{cases} a(k\rho^{*} - \rho_0^{*}), & k\rho^{*} \geqslant \rho_0^{*}; \\ 0, & k\rho^{*} < \rho_0^{*}. \end{cases} \tag{14.1}$$

The curves $p = p(\rho)$ in the case of $k = \text{const}$ consequently have a bend (Figure 46). As we may verify, this fact has a significant influence on several of the computational steps.

According to our definition expressed by equation (14.1), no regions of negative pressure may arise. Instead, regions of zero pressure appear, which

correspond to the motion of the destroyed substance. We shall call them regions of separation. In the region of separation, the motion satisfies the equations (1.19) - (1.20); only condition p = 0 must be taken into consideration.

However, it will be more advantageous for us to emply the equations of gas dynamics in the Lagrange representation, and we may select the quantity q equalling the mass M which is determined by equation (1.4) as the Lagrangian coordinate. /114

$$dq = \rho r^{\nu}(dr - u\,dt).$$

As is known, the Lagrange equations have the following form:

$$\left.\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) - \frac{\partial (r^{\nu}u)}{\partial q} &= 0, \\ \frac{\partial u}{\partial t} + r^{\nu}\frac{\partial p}{\partial q} &= 0, \\ \frac{\partial s}{\partial t} &= 0, \\ \frac{\partial r}{\partial t} &= u. \end{aligned}\right\} \qquad (14.2)$$

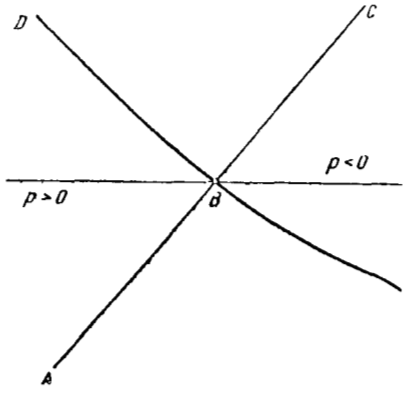

Figure 47

It follows from (1.4) that $\rho = r^{-\nu}\frac{\partial q}{\partial r}$ along the cross section t = const. Therefore the equation of discontinuity may be written as follows

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = -\left(\nu\frac{u}{r} + \rho r^{\nu}\frac{\partial u}{\partial q}\right). \qquad (14.3)$$

We should also point out that relationship (1.4) may be rewritten as follows

$$dr^{\nu} = \frac{\nu}{r\rho} dq + \nu u r^{\nu-1} dt,$$

and therefore

$$\frac{\partial r^{\nu}}{\partial q} = \frac{u}{r\rho}.$$

It follows from the Euler equation [the second equation of system (14.2)] that in the separation region, where p = 0, the particles of the substance move independently of each other with constant velocities along rectilinear trajectories. Integrating the last equation of system (14.2), we obtain /115

$$r - ut = f(u), \tag{14.4}$$

where f(u) is an arbitrary function.

In order to determine the density ρ, let us differentiate (14.4)

$$dr - u\,dt = (t + f')\,du.$$

The prime sign designates differentiation with respect to u. Substituting this expression in (1.4), we obtain

$$dq = \rho r^{\nu}(t + f')\,du. \tag{14.5}$$

The right part may be an exact differential only when pr^{ν} (t + f') depends only on u, i.e.,

$$\rho r^{\nu}(t + f') = F(u).$$

We thus have

$$\rho = \frac{F(u)}{r^{\nu}(t + f')}, \tag{14.6}$$

where F(u) is a new arbitrary function. Since the velocity u is a function of q (or R), in the separation region

$$s = s(u). \tag{14.7}$$

Formulas (14.4), (14.6) and (14.7) provide the total solution for the equations of motion in the separation region; therefore, numerical integration is superfluous here. The arbitrary functions f(u), F(u) and S(u) must be determined at the moment the separation is formed. Let us now discuss this step.

Let us first assume that we have calculated a certain region according to

the equation of state (5.15), and let us assume that as a result we have arrived at a negative value of p. Let DBE (Figure 47) be the line on which p = 0. The separation must develop in the region lying above this line.

Let us investigate the trajectory ABC of a certain particle. The BC portion of the trajectory will lie in the separation region; in particular, ρ is determined by formula (14.6). It is apparent that the separation can be formed only if, according to (14.6), ρ continues to drop after passing through the point B. We may now show that in order to do this it is necessary and sufficient that the direction of the line p = 0 be spacelike at the point B.

In the separation region u is a function only of q. Therefore, the /116 derivative of the function ρ with respect to time along the trajectory may be written, according to (14.3), in the following form

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = -\left(\nu\frac{u}{r} + \rho r^{\nu}\frac{du}{dq}\right). \tag{14.8}$$

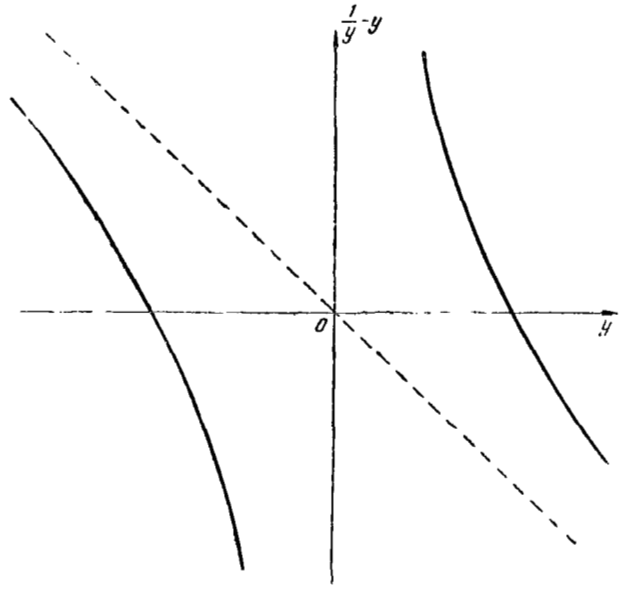

Figure 48

For the formation of the separation, it is necessary and sufficient that the right part in (14.8) be negative or, which is the same thing, that the expression in the parentheses be positive. This expression may be computed by taking the quantities u, ρ, r at point B and the derivative $\frac{du}{dq}$ by differentiation along

the p = 0 line - employing the functions in the region outside of the separation.

Let dq = λ dt hold along the curve p = 0. Then $\frac{du}{dq}$, and consequently the derivative $\frac{\partial \rho}{\partial t}$, can be expressed by partial derivatives in the region outside of the separation according to the following formulas

$$\frac{du}{dq} = \frac{\partial u}{\partial q} + \frac{1}{\lambda}\frac{\partial u}{\partial t} \tag{14.9}$$

and

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = -\left(\nu\frac{u}{r} + \rho r^{\nu}\frac{\partial u}{\partial q} + \frac{\rho r^{\nu}}{\lambda}\frac{\partial u}{\partial t}\right). \tag{14.10}$$

Let us express $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial q}$ along the p = 0 line by $\frac{\partial p}{\partial q}$ on this line. /117

It follows from the second equation of the system (14.2) that

$$\frac{\partial u}{\partial t} = -r^{\nu}\frac{\partial p}{\partial q}. \tag{14.11}$$

Since the following equation holds along the p = 0 line

$$\frac{\partial p}{\partial t} + \lambda\frac{\partial p}{\partial q} = 0, \tag{14.12}$$

by employing the 3rd and 1st equations of system (14.1) and (14.4), we obtain the following relationship between $\frac{\partial u}{\partial q}$ and $\frac{\partial p}{\partial q}$ along the p = 0 line:

$$-\lambda\frac{\partial p}{\partial q} = \frac{\partial p}{\partial t} = -c^2\rho^2\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = -c^2\rho^2\frac{\partial (r^{\nu}u)}{\partial q} =$$
$$= -c^2\rho^2\left(r^{\nu}\frac{\partial u}{\partial q} + u\frac{\partial r^{\nu}}{\partial q}\right) = -c^2\rho^2\left(r^{\nu}\frac{\partial u}{\partial q} + u\frac{\nu}{\rho r}\right).$$

Substituting the expressions obtained for $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial q}$ in (14.10), we obtain the value of the derivative $\frac{\partial p}{\partial t}$ along the trajectory in the separation region as follows

$$\frac{c}{\rho}\frac{\partial \rho}{\partial t} = \left(\frac{c\rho r^{\nu}}{\lambda} - \frac{\lambda}{c\rho r^{\nu}}\right)\frac{\partial p}{\partial q}r^{\nu}. \tag{14.13}$$

Let us designate $\frac{\lambda}{c\rho r^{\nu}}$ by y. The expression in parentheses is then $\frac{1}{y} - y$.

Figure 48 presents a graph of this function. An examination of this graph readily shows that $\frac{\partial p}{\partial t} < 0$ is fulfilled under the following conditions:

$$\left.\begin{array}{l} (1) \quad \text{if } \frac{\partial p}{\partial q} < 0, \text{ then either } y < -1 \text{ or } 0 < y < 1, \\ (2) \quad \text{if } \frac{\partial p}{\partial q} > 0, \text{ then either } y > 1, \text{ or } -1 < y < 0. \end{array}\right\} \tag{14.14}$$

In actuality, in the first case only the inequality $y < -1$ can hold, and in the second case - $y > 1$ - because the inequality $\frac{\partial p}{\partial q} < 0 \left(\frac{\partial p}{\partial q} > 0\right)$ means that the region outside of the separation lies to the left (to the right) and below the $p = 0$ curve in the (q, t) plane - i.e., $\lambda = \frac{dq}{dt} < 0$ (> 0). Since $c\rho r^{\nu} > 0$, y must have the same sign. According to (1.4),

$$\lambda = \frac{dq}{dt} = \rho r^{\nu}\left(\frac{dr}{dt} - u\right).$$

We thus have /118

$$y = \frac{\lambda}{c\rho r^{\nu}} = \frac{1}{c}\left(\frac{dr}{dt} - u\right)\begin{cases} < -1 & \text{in the first case,} \\ > 1 & \text{in the second case.} \end{cases} \tag{14.15}$$

Consequently,

$$\left.\begin{array}{ll} \frac{dr}{dt} < u - c = \beta & \text{in the first case,} \\ \frac{dr}{dt} > u + c = \alpha & \text{in the second case.} \end{array}\right\} \tag{14.16}$$

This means that the $p = 0$ line at a given point is spacelike, which must be proven.

Let us now assume that as a result of the computation with negative pressure we have obtained the $p = 0$ line, which is located as is shown in Figure 49 with respect to the grid of the characteristics. To the right of point A this line is spacelike; to the left of point A it is timelike. According to the statements presented above, the separation develops to the right of point A beyond the $p = 0$ line; there is no separation to the left of point A. Consequently, the AC line must exist, which delineates the separation region to the left; to the left of this line $p > 0$ will hold everywhere. Thus, the quantities within the angle BAC differ from those which would be obtained with allowance for the separation. This region is subject to recalculation. In concrete terms, this must be expressed in the fact that β-characteristics - which enter this region through the AC line - must carry with them the Reimann invariants reflecting the influence of the separation in the region lying to the right of the AC line.

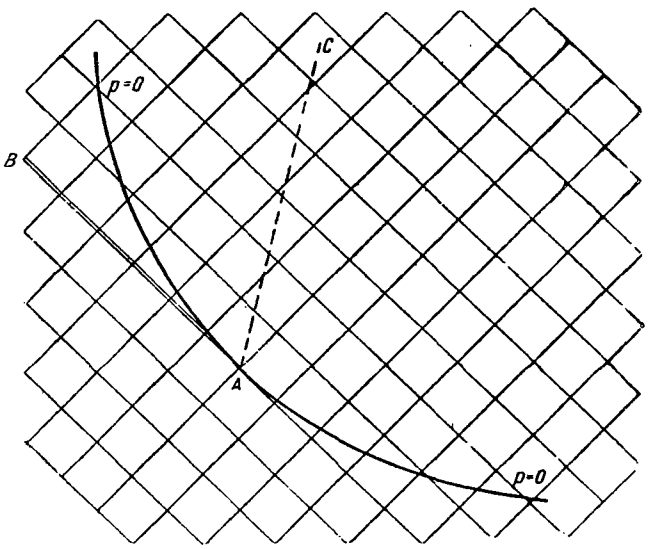

Figure 49

However, there are no α- and β-characteristics in the separation /119 region. The particles move independently of each other, and all of the interaction is transmitted only along the trajectory. In order that the effect indicated above may be manifested, this means that it is necessary for the particle trajectories, which intersect the line p = 0 to the right of point A, to also intersect the line AC. In other words, there must be a flux of matter from right to left through the line AC.

There is no basis for assuming that all of the quantities change continuously through the line AC. Formula (14.6) shows that the density ρ on the trajectory within the separation is determined exclusively by the initial data on the p = 0 line; when the trajectory reaches line AC, ρ upon it may have any magnitude. To the left of the line AC, $p \geqslant 0$ must hold which - according to the equation of state (14.1) - imposes specific limitations on ρ.

All of these statements show that the line AC is a shock wave. We shall now derive the relationships for the front of this wave (it is apparent that the formulas which we obtained previously are unsuitable; the lower quantities do not satisfy our customary equation of state). We may start with equations (9.13), which we obtained directly from the laws of conservation. Let us commence with the derivation of the Hugoniot adiabatic equation.

The expression for the internal energy of our substance is known [see (11.14)]:

$$\varepsilon = \frac{p + \rho_0 c_0^2}{(\varkappa - 1)\rho}. \tag{14.17}$$

However, this formula is not applicable within the separation region. The particles do not interact in this region; the internal energy of each particle remains constant. It immediately follows that in the separation region ε retains a constant value along the particle trajectory, and does not depend, in particular, on density ρ.

On the $p = 0$ line, we shall have

$$\varepsilon = \frac{\rho_0 c_0^2}{(\varkappa - 1)\rho},$$

or, according to (5.21), we have

$$\varepsilon = \frac{v^2}{\varkappa - 1} \tag{14.18}$$

(since $p = 0$, then $z = 1$). Since both v and ε remain constant during subsequen motion along the trajectory, (14.18) remains valid within the entire separation region.

The third relationship (9.13) may now be written as follows /120

$$\frac{p_2 + \rho_0 c_0^2}{(\varkappa - 1)\rho_2} - \frac{v_1^2}{\varkappa - 1} = -\frac{p_2}{2}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right).$$

Let us set $\sigma = \frac{\rho_2}{\rho_1}$. After simple computations, we obtain the following Hugoniot adiabatic equation,

$$\sigma = \frac{hp_2 + (h-1)\rho_0 c_0^2}{p_2 + (h-1)\rho_1 v_1^2}. \tag{14.19}$$

Substituting z for p, according to (5.19), we obtain

$$\sigma = \frac{hz_2^{h+1} + 1}{z_2^{h+1} + (h+1)\frac{\rho_1 v_1^2}{\rho_0 c_0^2} - 1}. \tag{14.20}$$

It follows from (5.21) that

$$\sigma = \frac{\rho_2}{\rho_1} = \frac{\rho_0 c_0^2}{\rho_1} \cdot \frac{z_2^{h-1}}{v_2^2}.$$

Therefore, we have the following from (14.20)

$$v_2^2 = \frac{c_0^2}{\delta_1} z_2^{h-1} \frac{z_2^{h+1} + (h+1)\frac{\delta_1 v_1^2}{c_0^2} - 1}{h z_2^{h+1} + 1}, \tag{14.21}$$

where $\frac{\rho}{\rho_0}$ is designated by δ.

We may write the second equation (9.13) in the following form

$$(u_2 - u_1)^2 = -\frac{p_2}{\rho_1}\left(\frac{1}{\sigma} - 1\right),$$

and by means of (14.20) we may readily obtain

$$(u_2 - u_1)^2 = \frac{c_0^2}{\varkappa\delta_1} \frac{(z_2^{h+1} - 1)\left[(h-1) z_2^{h+1} - (h+1)\frac{\delta_1 v_1^2}{c_0^2} + 2\right]}{h z_2^{h+1} + 1}. \tag{14.22}$$

Finally, the first relationship (9.13)

$$D - u_1 = \frac{\sigma}{\sigma - 1}(u_2 - u_1)$$

may be reduced to the following form

$$D - u_1 = \frac{h z_2^{h+1} + 1}{(h-1) z_2^{h+1} - (h+1)\frac{\delta_1 v_1^2}{c_0^2} + 2}(u_2 - u_1). \tag{14.23}$$

Collecting the formulas (14.21) - (14.23) all together, we obtain a system of relationships for our shock wave /121

$$\left.\begin{aligned}
(u_2 - u_1)^2 &= \frac{c_0^2}{\varkappa\delta_1} \frac{(z_2^{h+1} - 1)\left[(h-1) z_2^{h+1} - (h+1)\frac{\delta_1 v_1^2}{c_0^2} + 2\right]}{h z_2^{h+1} + 1}, \\
v_2^2 &= \frac{c_0^2}{\delta_1} z_2^{h-1} \frac{z_2^{h+1} + (h+1)\frac{\delta_1 v_1^2}{c_0^2} - 1}{h z_2^{h+1} + 1}, \\
D - u_1 &= \frac{h z_2^{h+1} + 1}{(h-1) z_2^{h+1} - (h+1)\frac{\delta_1 v_1^2}{c_0^2} + 2}(u_2 - u_1).
\end{aligned}\right\} \tag{14.24}$$

Equations (14.24) are completely similar to equations (11.6); the shock wave may be computed by the method presented in Section 11.

Thus, the separation may be computed in the following way.

After a pressure decrease below zero has been detected, the line p = 0 must be determined on the r, t plane. This is done by a small computation in the region of negative pressures, with subsequent interpolation along the characteristics for the value p = 0 (or, which is the same thing, for z = 1). The quantities r, t, u, v, δ, R are determined at points on this line.

There is a spacelike segment on the line p = 0. The shock waves enclosing the separation will emanate from the ends of this segment.

The quantity f = r - ut may be computed at all points on this segment. A table may be compiled, whose argument is u, and whose function is f. Then, by numerical differentiation according to this table, we may obtain the derivative $f' = \frac{df}{du}$ and may calculate the following quantity

$$F = \delta r^{\nu}(t + f');$$

The values of f' and F at each point on the initial segment of the p = 0 line are written in the third and fourth columns of this table. The fifth column contains v and the sixth column contains the Lagrangian coordinate R.

After this table is prepared, the shock waves enclosing the separation may be computed. As a rule, they have zero amplitude at the initial point, and the quantity D remains indeterminate in contrast to "normal" waves. This may be explained by the discontinuity in the equation of state (see Figure 46). The derivative $\frac{dp}{\partial\rho}$ does not exist at the discontinuity point, and therefore the speed of sound c is indeterminate.

We must set D = u at the initial point. It is recommended that the first step along the wave front line not be too small, so that a fairly accurate /122 determination of D may be made at the first computed point.

After the coordinates of the subsequent point on the front are determined, it is necessary to find the lower quantities u_1, v_1, ρ_1. The quantity u_1 may be found from the equation

$$r - ut = f(u)$$

by means of the table compiled. Substituting the values of r, t, which have been found, in this equation we may determine u by the customary numerical solution of the equation, employing the table of the function f(u). In several cases, this function may be quite accurately approximated by the line

$$f = au + b.$$

Then the equation may be clearly solved as follows:

$$u = \frac{r - b}{a + t}.$$

It must be noted that this approximation is advantageous if the entire separation region may be encompassed by one linear function. If discontinuities are produced in the function f(u) as the result of the "piecewise" approximation - i.e., discontinuities of the derivative - this leads to discontinuities of δ_1, and the decays of these discontinuities must be computed at the corresponding points on the enclosing wave front.

After u_1 is determined, we may find f' and F from the same table and may determine δ_1:

$$\delta_1 = \frac{F}{r^{\nu}(1 + f')}.$$

We may then find v_1 and R_1 directly from the table (R_1 is necessary to determine $R_2 = R_1$). The subsequent computation of the wave point proceeds in the normal manner.

If a normal shock wave "overtakes" the wave enclosing the separation, the arbitrary discontinuity which is produced may be computed in the normal manner. Usually when the shock wave "cuts in" into the separation region, it leads to the rapid enclosure of it. Naturally, expansion waves have the opposite effect.

15. INTEGRAL CONTROL /123

In Section 4 we have already discussed the different methods for controlling the accuracy of numerical integration. We then discussed "local" control and a determination of the error within the limits of one or several steps. We shall now discuss determining the accuracy of the solution for the entire problem as a whole, when the main computations have already been performed. Such control may be conveniently exerted by verifying that the laws of conservation have been fulfilled. This may best be achieved by employing their integral form (1.19). We shall make a separate determination of each method to be employed in verifying that the laws of conservation of mass, momentum, and energy have been fulfilled.

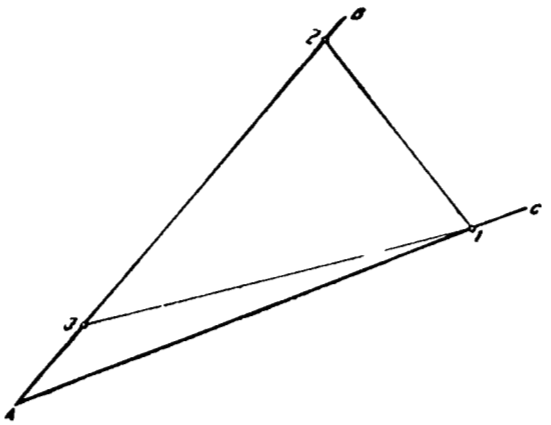

Figure 50

The conservation of mass may be checked by means of the Lagrangian coordinate R. We already discussed in Section 4 the rational selection of the integration direction of R. We shall now investigate this problem from another viewpoint.

Let us commence with the following simple example. We shall assume that we are investigating the passage of a shock wave through a gas at rest under the influence of a plunger which moves in a definite way. Let the line AB (Figure 50) express the movement of the plunger on the r, t plane, and the line AC - the shock wave front. The quantity R retains a constant value on /124 the AB line; we can set R = r on the AC line. Let us select any β-characteristic 1 - 2, and - after this characteristic is computed entirely - let us integrate R along it from point 1 to point 2 employing the quantities R' which have already been calculated at every point of this characteristic:

$$R_2 - R_1 = -\int_1^2 R' dt. \qquad (15.1)$$

We may find the same difference by another method. In actuality, we knew R_2 beforehand, and $R_1 = r_1$. The difference between these two results enables us to make a specific determination of the over-all computational accuracy within a triangle with the apexes at points A, 1 and 2. We may introduce a certain quantitative determination of this accuracy. Setting $R_1 = r_1$ in formula (15.1), we obtain R_2 from it, which we may designate by R_2^* in contrast to the "exact" value of the same quantity $R_2 = R_A$. The equation

$$\delta R = \frac{R_2 - R_2^*}{R_2 - R_1} \qquad (15.2)$$

may serve as a measure of the accuracy with which the law of conservation of mass has been fulfilled. Since the mass is proportional to R^ν, we may employ another quantity in addition to δR:

$$\delta R^* = \frac{R_2^\nu - R_2^{*\nu}}{R_2^\nu - R_1^\nu}. \qquad (15.3)$$

Both of these quantities δR and δR^ν may be advantageously expressed in percents.

It is impossible to draw any simple connection between δR and δR^ν with the errors of our computational results. On the other hand, the requirement that δR^ν does not exceed 1% represents a definite requirement for the over-all computational accuracy. In exactly the same way, if we had the computational results for two problems, and if the control of the conservation of mass described above provides a quantity on the order of 0.5% for δR^ν in the first case - and a quantity on the order of 5% in the second case - then we may state with a great degree of certainty that the first computation is significantly more accurate than the second.

From the very beginning, if we agree to integrate the Lagrangian coordinate R along the β-characteristics in this problem, we avoid the necessity of making a separate computation of the integral (15.1), and R_2* will automatically be obtained at each point on the boundary AB. It is then possible for us to trace the changes in the determination of δR and δR^ν during the computational process, and to employ the appropriate measures (for example, to decrease the step) concurrently, in the case of their systematic increase. If inte- /125 gration of R along the β-characteristics is disadvantageous (for example, due to the rapid change in R' in this direction), we may similarly determine the fulfillment of the law of conservation of mass along the α-characteristics (3 - 1 in Figure 50). We shall compare $R_1 = r_1$ with R_1* calculated by integration

$$R_1^* = R_3 + \int_3^1 R' dt.$$

The quantities δR and δR^ν may be computed in the following way:

$$\delta R = \frac{R_1 - R_1^*}{R_1 - R_3}, \quad \delta R^\nu = \frac{R_1^\nu - R_1^{*\nu}}{R_1^\nu - R_3^\nu}.$$

It may happen that the problem is to calculate the line of the shock wave front AC up to point 1 and the boundary AB up to point 2. When the computation is completed, the integral (15.1) must also be computed. If the nature of the change in R' does not enable us to trace it with sufficient accuracy based on the formula in the table, it is then advantageous to employ a more precise quadratic formula - for example, (4.1).

Finally, it is possible that it may be advantageous to compute the Lagrangian coordinate R along the α-characteristics in one section of the region under consideration, and along the β-characteristics in another section. The control of the law of conservation may then be exerted on the boundary of these sections (Figure 51, the boundary of the sections is designated by the dashed line). Two values of the Lagrangian coordinate are obtained at point 3.

$$R_3^* = R_2 + \int_2^3 R' dt, \quad R_3^{**} = R_1 - \int_1^3 R' dt.$$

The "accurate" values of R_2 and R_1 are chosen - the constant value of R_2 on the boundary and $R_1 = r_1$ on the shock wave. We then obtain

$$\delta R = \frac{R_3^{**} - R_3^*}{R_1 - R_2},$$

$$\delta R^\nu = \frac{R_3^{**\nu} - R_3^{*\nu}}{R_1^\nu - R_2^\nu}.$$

The region between the two contact discontinuities may be controlled in a similar way. If, for example, R is computed along the α-characteristic, then a value corresponding to the right discontinuity is attributed to the Lagrangian coordinate R at point 1 (Figure 52). We shall then have two values at point 2 - the "accurate" value corresponding to the right discontinuity, and the value obtained as a result of integration along the 1 - 2 characteristic. /126
Just as previously, we may compute δR and δR^ν and may employ them to determine the accuracy of the conservation of mass in the given region.

It is also recommended that the Lagrangian coordinate R be calculated for the shock waves by integration along the front line. Since the equation of this line is dr = D dt, we obtain the following from (1.17)

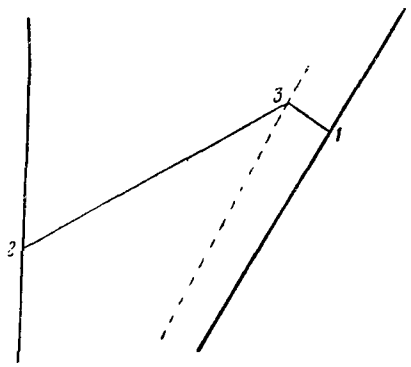

Figure 51

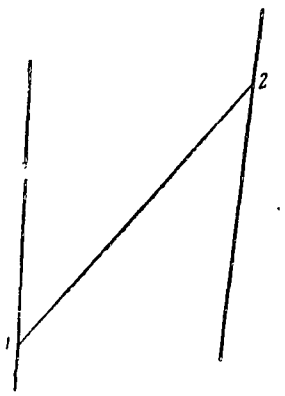

Figure 52

$$dR = \delta \left(\frac{r}{R}\right)^{\nu} (D - u)\, dt, \tag{15.4}$$

where, as always $\delta = \frac{\rho}{\rho_0}$. Formula (15.4) may also be rewritten in the following form

$$dR = \frac{D - u}{c} R' dt. \tag{15.5}$$

Employing these formulas, we may integrate R directly over the shock wave front. The values thus obtained may be compared with the values interpolated according to the lower data. When the shock wave reaches the contact discontinuity, the corresponding comparison may also be performed, and a definite conclusion may be reached regarding the accuracy with which the law of conservation of

mass is satisfied.

Leaving the law of conservation of momentum, let us turn to the law of conservation of energy. The quantity E, which may be determined by the third formula (1.18), may not be computed at each point, and therefore for a control of the conservation of energy we must integrate the following expression

$$dE = \rho\left(\varepsilon + \frac{u^2}{2}\right) r^\nu dr - u\left[\rho\left(\varepsilon + \frac{u^2}{2}\right) + p\right] r^\nu dt \tag{15.6}$$

separately, choosing a closed contour encompassing the region to be controlled as the integration path. It is frequently advantageous to express the energy differential in Lagrangian coordinates [see (1.22)]: /127

$$dE = \rho_0\left(\varepsilon + \frac{u^2}{2}\right) R^\nu dR - upr^\nu dt. \tag{15.7}$$

Let us express it by our "working" variables u, v, z.

For an ideal gas, we have the following on the basis of (11.1) and (5.2)

$$\varepsilon = \frac{c^2}{\varkappa(\varkappa - 1)}.$$

Substituting this expression in (15.7) and allowing for (5.11), we obtain

$$dE = \rho_0\left[\frac{c^2}{\varkappa(\varkappa - 1)} + \frac{u^2}{2}\right] R^\nu dR - \frac{a^2}{\varkappa} uz^{h+1} r^\nu dt. \tag{15.8}$$

In particular, dr = u dt along the trajectory, i.e., dR = 0

$$dE = -\frac{a^2}{\varkappa} uz^{h+1} r^\nu dt. \tag{15.9}$$

For an equation of state such as (5.15), the formula for internal energy has the form (11.14). In several cases, however, it is advantageous to determine that $\varepsilon = 0$ in the case of $p = 0$ and $\rho = \rho_0$, by adding a constant component to this formula. In order to do this, we must set

$$\varepsilon = \frac{p + \rho c_0^2}{(\varkappa - 1)\rho} - \frac{c_0^2}{\varkappa - 1}. \tag{15.10}$$

By means of formulas (5.19) and (5.21) we may now write

$$\varepsilon = \frac{c^2}{\varkappa(\varkappa - 1)} + \frac{v^2}{\varkappa z^{h-1}} - \frac{c_0^2}{\varkappa - 1}, \tag{15.11}$$

and the energy differential assumes the form

$$dE = \rho_0 \left[\frac{c^2}{\varkappa(\varkappa-1)} + \frac{u^2}{2} + \frac{v^2}{\varkappa z^{h-1}} - \frac{c_0^2}{\varkappa-1} \right] R^\nu dR - \frac{\rho_0 c_0^2}{\varkappa} u(z^{h+1}-1) r^\nu dt. \tag{15.12}$$

In particular, we have the following along the trajectory

$$dE = -\frac{\rho_0 c_0^2}{\varkappa} u(z^{h+1}-1) r^\nu dt. \tag{15.13}$$

We should point out that we can transform formula (15.12) so that it formally coincides with (15.8). In order to do this, we may write (15.11) in the following form (for purposes of convenience we shall discard the constant component)

$$\varepsilon = \frac{c^2}{\varkappa(\varkappa-1)} + \frac{c_0^2}{\varkappa\delta}.$$

And we now readily obtain /128

$$dE = \rho_0 \left[\frac{c^2}{\varkappa(\varkappa-1)} + \frac{u^2}{2} \right] R^\nu dR - \frac{\rho_0 c_0^2}{\varkappa} u z^{h+1} r^\nu dt + \frac{\rho_0 c_0^2}{\varkappa} \left(\frac{R^\nu dR}{\delta} + u r^\nu dt \right)$$

However, formula (1.21) shows that the expression in the parentheses is $r^\nu dr$. Therefore, the third component may be written as follows

$$d\left[\frac{\rho_0 c_0^2}{(\nu+1)\varkappa} r^{\nu+1} \right].$$

Since r is everywhere continuous, during integration over any closed contour this differential yields zero. Consequently, for a control of the law of conservation of energy we may also employ the following quantity, in addition to E

$$E^* = E - \frac{\rho_0 c_0^2}{(\nu+1)\varkappa} r^{\nu+1}.$$

For its differential, we have

$$dE^* = \rho_0 \left[\frac{c^2}{\varkappa(\varkappa-1)} + \frac{u^2}{2} \right] R^\nu dR - \frac{\rho_0 c_0^2}{\varkappa} u z^{h+1} r^\nu dt. \tag{15.14}$$

It must be stipulated that formula (15.14) may be employed advantageously only if we are dealing with one substance. Otherwise, E* will have a different meaning in different regions (in view of the difference ρ_0, c_0 and $\varkappa$), an integration of its differential along the contact discontinuity dividing the two different substances will lead to different results - depending on whether we select the right or left values for the integrands (E* ungoes a discontinuity on such contact discontinuities). This entails additional difficulties, and

therefore formula (15.12) must primarily be employed.

The law of conservation of energy may be suitably controlled as follows. Let us delineate on the r, t-plane the region included between two contact discontinuities (as a rule, between the boundaries separating different substances). Let the computation of this region begin with $t = t_0$ and conclude at $t = t_1$ (see Figure 53). By integration over the lines $t = t_0$ and $t = t_1$ (in both cases - in the direction of an R increase) we may determine E_0 and E_1 of the total energy included within the layer we have selected at the moments t_0 and t_1. By integration along the layer boundaries (in the direction of an increase in t), we may then find the work A_1 and A_2 of the pressure forces at these boundaries. According to the law of conservation of energy the following must hold

$$E_1 - E_0 = A_2 - A_1.$$

The difference between the right and left parts of this equality, ob- /129
tained as a result of the integration, may serve as a measure of the accuracy with which the law of convervation of energy is satisfied.

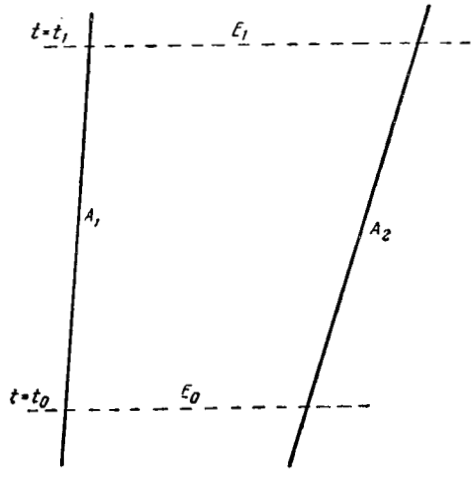

Figure 53

However, this quantity still does not present an idea of the accuracy of the problem's solution. It would be desirable to have a certain relative determination which would make it possible to express the distribution of the balance of energy in percents. In addition, it would be advantageous to require an "additive property" from this determination, which would be as follows. Let us divide the region to be controlled by a certain line into two parts, and let us perform the computation in each part separately. It is necessary that the determination pertaining to the entire region as a whole be included between the

determinations pertaining to each part separately.

We can make such a determination as follows. First of all, let us stipulate that during the integration we always go around any contour in the counterclockwise direction. We shall integrate the differential (15.7) over this contour, beginning with a certain point where we shall set E = 0. The quantity E will increase on certain sections and will decrease on other sections, when moving in the direction we have selected. Let us calculate the integral I_+ separately over all the sections of an increase, and let us calculate I_- over all the sections of a decrease. Their sum will be close to zero. Let us now set

$$\delta E = \frac{I_+ + I_-}{I_+}. \tag{15.15}$$

As may be readily seen, this quantity has all of the requisite features. We shall select it as the determination of the accuracy with which the law of conservation is satisfied.

Let us now turn to Figure 53. Let us assume that both integrals E_0 and E_1 are positive (as will be case in the overwhelming majority of cases), and that the integrals A_1 and A_2 are negative (which will hold if u > 0, p > 0 along both boundaries). We may perform the calculation as follows

$$\delta E = \frac{(E_0 - A_1) - (E_1 - A_2)}{E_0 - A_1}.$$

If A_1 and A_2 are positive, then /130

$$\delta E = \frac{(E_0 + A_2) - (E_1 + A_1)}{E_2 + A_2}.$$

In every other case, we must turn directly to the rule formulated above (15.15).

If the motion is initiated from a state of rest, so that u = 0, p = const, ρ = const will hold in the case of $t = t_0$, the computation of E_0 is then significantly simplified. For example, for an ideal gas we have

$$E_0 = \frac{\rho_0 c^2}{(\nu+1)\,\varkappa\,(\varkappa-1)}\left(R_2^{\nu+1} - R_1^{\nu+1}\right).$$

If p = 0 then $E_0 = 0$. Just as for a substance of the type (5.15), $E_0 = 0$ is obtained in the case of p = 0, $\rho = \rho_0$ [this was the reason for introducing a constant component in formula (15.10)]. The upper integration path (as well as the lower) need not be chosen in the form of a horizontal line; it may be any line defining the region to be calculated above. The region to be controlled may contain as many shock waves and contact discontinuities as desired. It is expedient not to control the problem as a whole, but in separate layers, especially if these layers differ greatly in terms of their physical properties.

We must now investigate the problem of verifying the law of conservation of momentum. In the case of flat, one-dimensional motion ($\nu = 0$), the control of the solution by this law does not differ, in essence, from the control over conservation of energy, and we shall not discuss it. With respect to cases of cylindrical and spherical symmetry, very serious difficulty is entailed. This difficulty is related to the fact that the integral law of conservation (1.10) contains an area integral. The calculation of these integrals entails very time-consuming work; it is valid only in particularly important cases. The verification of the conservation of momentum may not therefore be regarded as a "working" method for controlling the accuracy with which the problem is solved. In view of this, we shall not investigate this problem.

Both methods of integral control presented above may serve for verification of the over-all accuracy of numerical integration, and for the determination of computational errors. It must be only kept in mind that they cannot reveal all of the errors. For example, they cannot reveal such errors as errors in equations of state, boundary conditions, et cetera - in short, errors designating the substitution of one problem by another. The validity and accuracy of the solution for the problem may be adequately verified only if all of the control methods which have been examined are combined during the process: both local methods and integral methods.

If an abnormally large disturbance of the balance is found when the /131 conservation of energy is checked in any region, and the local control methods (recalculations, et cetera) provide no basis for explaining this fact by a great amount of error in the numerical integration, it must then be assumed that there is a computational error. In order to discover it, the region to be studied may be divided into two portions, and each portion may be controlled individually. The portion which contains the error must be subdivided further, et cetera. In this way, it is almost always possible to localize the error comparatively rapidly, and to completely expose it.

APPENDIX 1. PROGRAMMING PRINCIPLES

1. GENERAL REMARKS /132

The tendency to employ electronic computers for purposes of maximum mechanization and automation in solving the problems we are investigating is fully understandable, and necessitates no special justification. However, the programming of the characteristics method presented here entails several difficulties related primarily to the comparatively great logical complexity of this method. We do not know of even one programming variation in which all of these difficulties may be overcome. Nevertheless, we feel it is advantageous to present a brief description of one of the variations. It is understood that the programs to be described are far from perfect. However, on the one hand,they may be recommended for use in practice, and on the other hand they may serve as a basis for the development of new, more refined variations. For fully understandable reasons, we shall not discuss all of the logical and computational details, but shall confine ourselves to only a very general description.

The principles for compiling a program depend on the structure of that class of problems for whose solution they are intended. If, for example, we are dealing with problems of a strictly determined structure (for example, that shown in Figure 49), then the programming encounters very few fundamental difficulties. However, we are interested in programs which could be regarded as having universal application. Thus, different problems may contain a different number of discontinuities -- contact discontinuities and shock discontinuities; the substances may have different types of equations of state; the boundary conditions may change from problem to problem, etc. The programs must thus be compiled in such a way that they may be applied to different problems without entailing any changes.

When speaking of the universal application of a program, we are employing the term universal application in a relative sense -- the variety of imaginary types of problems is infinite, and the attempt to produce an absolutely universal program is doomed to failure. We must now introduce any limitations on the structure of the problem. With respect to the programs which will be described here, the possible class of problems will be characterized in detail below. We would like only to point out that the solution of this class /133 of problems must be regarded as one of the most real problems. Every new program must first be evaluated from this point of view.

It is also clear that even under the limitations discussed above, the "universal" program must inevitably be of such a great volume that it cannot be placed in the operative memory of the types of machines presently in use. Therefore, we can only speak of producing a collection or set of programs which - when operating in a specific order - could replace each other in the operative memory. Before the initial computation, all of these programs must be written in the outer memory (for example, on a magnetic drum), and may

then be transferred to the operative memory when needed. On the one hand, this imposes certain demands on the construction of the computer and, on the other hand, creates specific conditions for greater flexibility during the operation of the program. Although the complete set of programs may be extremely extensive, for a specific, given problem it is only necessary to select those programs which are necessary in the given case.

Before the beginning of the computation, the total structure of the given problem must be coded in some manner, and be introduced into the machine together with the initial data. The sequence of calls and the operational regime of the separate programs must be determined by this information regarding the structure of the problem. Thus, in addition to purely computational functions, the programs must also first fulfill specific logical functions. Each program must "know" at the moment it is called the order in which it must operate, and which program it must call after it has finished. Since each program must be designed to operate under very different conditions, the task of interpreting the coded information regarding the structure of the problem and of determining the operational order of the program will present a very serious problem.

This problem was solved by introducing special logical programs. Thus, the programs are divided into computational and logical programs. The computational programs fulfill all of the basic computational work; their logical functions are minimal. These programs include programs for calculating a part of the characteristics, the point of the contact discontinuity, the point of the shock wave, etc. When they are called into the operative memory, they always perform one and the same computation according to the established scheme, independently of the over-all structure of the problem. They select the initial data from the locations indicated to them; the results are located also in the indicated locations. When the operation is completed, they call the programs indicated to them previously.

The logical programs do not perform any "useful" computations. /134 Their function is reduced to controlling the computational program and to insuring their normal operation. They determine the sequence of calls for the computational programs, when necessary they include in the cycle - or on the other hand, exclude from the operation - the individual computational programs, they prepare the initial data, they distribute the results, etc. The operational order of the logical programs depends entirely on the structure of the problem.

2. GENERAL SEQUENCE OF THE COMPUTATION

The programs which we shall discuss were designed for the "Strela-1" machine; therefore, we should give a brief description of this machine before the following discussion.

The operative memory of the machine which is comprised of cathode-ray tubes contains 2047 elements, each of which contains one 43-digit binary code. In addition, there is a magnetic drum (about 5000 elements) and an almost unlimited outer memory on magnetic tapes.

The numbers in the memory of the machine are presented in a binary number system as a floating point decimal. Thirty-five binary digits are assigned to the mantissa, which corresponds to approximately ten decimal digits. The numerical characteristic occupies six binary digits. The machine can operate with numbers lying in between (approximately) from 10^{-19} to 10^{19} (with respect to the modulus).

There is a three-address system of commands; each address occupies 12 binary digits. Under the code of operation, 6 digits are removed. The remaining digit which is not used plays an auxiliary role.

The input and output devices employ standard 80-column punched cards for computers. The coding of the card punching for the input and output coincide, so that the cards coming out of the machine may be again introduced - when necessary - without interruption. The cards are printed on a separate device.

The sequence in which the problem is solved can be suitably analyzed by a specific example. Let us assume that the problem consists of computing the motion of two layers of substance separated by a contact discontinuity (Figure 54). Let us assume that certain boundary conditions are given to the right and to the left. The initial data are given at $t = t_0$. Let us assume that at this moment the left boundary (which was previously stationary), suddenly begins to move at a certain velocity (which may change subsequently). A diverging shock wave is produced at the moment t_0 at the left boundary. When it reaches the contact discontinuity, discontinuity decay occurs. Then the wave passes to the right boundary, from which it is reflected depending on the form of the shock wave or on the form of the expansion wave, as a function of the right boundary condition. Let us assume that from points A and B expansion /135 waves pass to the left. Let us assume that the purpose of the calculation is to determine the motion at the moment $t = t_1$.

Let us investigate the β-characteristic 1 - 1'. It begins on the line of the initial data ($t = t_0$), intersects the contact discontinuity and the shock wave, and terminates on the left boundary (point 1'). Let us place the point 2 which is quite close to point 1 on the line for the initial data. Based on the 1 - 1' characteristic and point 2, we may compute the segment of the subsequent β-characteristic (shown by the dashed line) up until it reaches the contact discontinuity. As may be readily seen, we may also calculate the point of the contact discontinuity line on the dashed characteristic, and may continue the calculation up to the shock wave front. Calculating the shock wave point, we extend a new characteristic up to the left boundary (point 2'). Returning to the line $t = t_0$, we may now take the subsequent point, and may compute the following characteristic in the same order, etc.

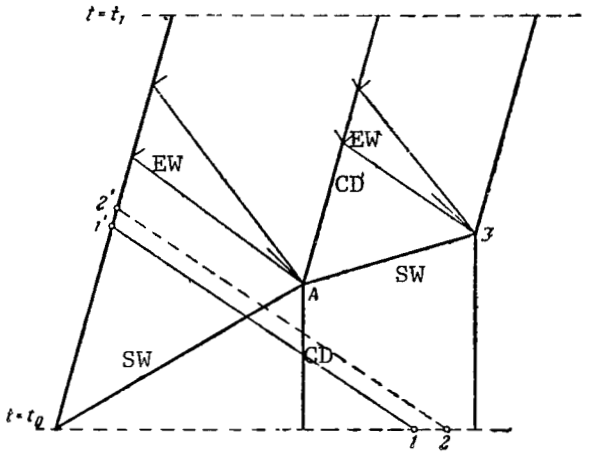

Figure 54

At the moment when the subsequent characteristic reaches the point A (i.e., when the shock wave reaches the contact discontinuity), the process must be terminated in order to compute the discontinuity decay. It is then necessary to calculate the region encompassed by the expansion wave, after which the computation may be continued in the same sequence. Beginning at a certain moment, the β-characteristics will take their origin from the right boundary, whose next point must be computed before the cycle is initiated. When the shock wave approaches the right boundary (point B), the discontinuity decay and the expansion wave must be computed again. After this, the problem may be solved up to the end, without any "interference."

If the problem contains converging shock waves, difficulties are then encountered due to the fact that such waves "truncate" the β-characteristic. The computation must then be done for the α-characteristics. /136

This is the manner in which the described programs operate. The calculation is performed only along the α- or along the β-characterstic; the automatic change from one regime to another is not specified. Therefore, the problem under consideration may contain only converging, or only diverging shock waves. This represents one of the very important conditions which limit the universal application of our programs.

The second limitation consists of the fact that the computation of discontinuity decays is not specified by the programs; in every case that different discontinuities are "encountered," the computation must end. In essence, the programs described are not designed for calculating the problem as a whole, but only for calculating individual regions which satisfy the above-mentioned conditions. The complex problem must be divided into sections which do not

contain discontinuity decays or shock waves of different directions. The discontinuity decays are computed manually. In order to simplify the programs, they do not include a change in the step during the computational process, so that the grid of characteristics is fixed - it is determined by the initial data. It is true that the programs control the recalculation quantities and the difference in the magnitude of the Lagrangian coordinate R computed along different characteristics. If these quantities exceed the given constants, the programs produce a stop with the corresponding signaling. In addition, the special features of the computation are not taken into account in any way by means of first-order discontinuities.

For purposes of determinancy, we shall only investigate a computation along the β-characteristics. As was already pointed out, the primary working cycle consist of computing the next β-characteristic. These characteristics appear in the memory of the machine in the form of a sequence of points, each of which consists of six quantities: t, r, R, u, z, v, and each of which occupies six elements of the memory. The points are arranged in the order of decreasing t and comprise the so-called main sequence. We shall call the address of the memory element containing t of this point the point address. Thus, the address of two adjoining points is distinguished by six units. The discontinuity points occur in two elements corresponding to two sides of the discontinuity. It is understood that t, r, R (and u for the contact discontinuity) coincide in both elements. During the process by which the new β-characteristic is computed, its points are recorded at the location of the points for the old characteristic, so that while the main cycle is being performed the main sequence contains the upper segment of the old characteristic (for the younger addresses) and the lower segment of the new characteristic (for the older addresses).

In addition, the initial data (cross section $t = t_0$ in Figure 54) are stored in the memory of the machine, also in the form of a sequence of points. It is also necessary to stipulate the location for the programs, which are subsequently read out from the magnet drum, with the next one taking the place of the preceding one. /137

The logical information on the structure of the problem, also reflecting the state of the computation at a given moment, is stored in the so-called controlling elements; their contents will be discussed at a later point. The different numerical constants of the problem (adiabatic indices for all substances, the quantities $_0$, c_0, etc.) are located here. Finally, there are two groups (first and second) of standard elements, with six elements in each group. They are designed for transmitting the numerical material from program to program.

3. COMPUTATIONAL PROGRAMS

Let us now describe the operation of the computational programs. We should first point out the following. As has already been indicated, the main

working cycle consists of computing the successive β-characteristic. Its points are subsequently computed in the order of increasing t and occupy the location of the points for the old β-characteristic in the main sequence. There are two addresses in the information stored in the controlling elements - the selection address and the recording address. The selection address coincides with the address of the next point for the old β-characteristic - the point which must be selected in order to compute the next point of the new characteristic. The recording address indicates where the last calculated point must be delivered. Naturally, both of these addresses change during the computational process (as a rule, they decrease by six units after each point is computed). Their initial state (before the next cycle) is established by logical programs. It is understood that at each moment the readings of these addresses must be matched, so that a new point which has just been computed does not occupy the position of the requisite point for the old characteristic. As may be readily shown, in order to do this it is necessary that the recording address be no less than the selection address.

Let us now investigate separate computational programs.

Program for computing a segment of the characteristic (or, as we shall designate it, the αβ program). The operational cycle of this program includes the computation of point 3 according to points 1 and 2 (Figure 55). Before the program is called, point 2 must be recorded in the second group of standard elements (this is performed by the preceding program). In addition, just as always, the selection and recording addresses must be indicated.

The operational cycle of the program consists of the following operations.

1. The point of the main sequence, whose address coincides with the selection address, is transferred to the first group of the standard elements. The selection address is decreased by six units.

2. A new point 3 is computed, according to the computational formulas in Section 5, based on the points included in the first and second groups of the standard elements. The results are stored in the working elements of the program. /138

3. A point from the second group of standard elements is transferred to the main sequence at the location indicated by the recording address. The recording address is decreased by six units.

4. Point 3 which was just computed is transferred from the working elements to the second group of standard elements.

5. A test of the segment end of the characteristic is conducted (see below). Depending on the results of this test, the program either repeats operation 1 - 5 or ends the operation and calls the subsequent program.

Thus, as may be readily seen, the segment of the new characteristic will

be computed until the test of the end (point 5) causes the computation to terminate.

The test of the segment end of the characteristic may be formulated in a different way. The program must discontinue the operation after the discontinuity point or the last point of the old characteristic is transferred to the first group of standard elements. The simplest procedure is to supply such points with a certain sign, on the basis of which the program can distinguish them. For example, it is possible to proceed as follows. In the cylindrical and spherical problems, r is always positive; in the plane problem, it is always possible to select the origin so that $r > 0$ also holds. Let us assign a minus sign to the "special" points (i.e., the discontinuity points and the last point of the characteristic) for r, when these points are in sequence. Immediately after the next point of the main sequence is transferred to the first group of standard elements, the αβ program tests the sign of r, which will constitute a test of the segment end of the characteristic. Before the computation of the next point of the new segment of the characteristic, the minus sign for r in the first group of standard elements must be canceled. In addition, it is necessary that each time, when any program stores a "special" point in the main sequence, r of this point assumes a minus sign.

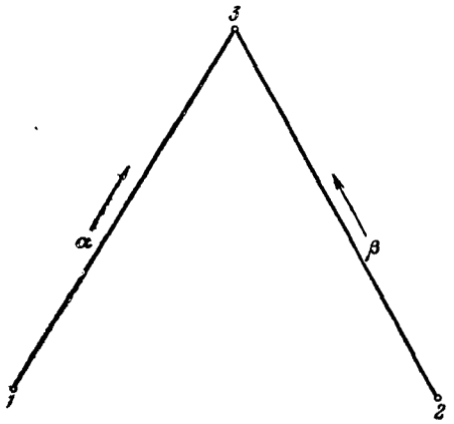

Figure 55

We should point out that when the operation of the αβ program has terminated, the last computed point remains in the second group of standard cells and is not stored in sequence. The last point employed of the old characteristic is included in the first group of standard elements. /139

Program for computing the contact discontinuity point (or, more precisely, CD program). As a rule, this program operates after the program αβ and also calls the program αβ after it is finished (generally speaking, another program - i.e., pertaining to another region). Its operational sequence is as follows (Figure 56). At the moment point 4 is called, which is the last calculated

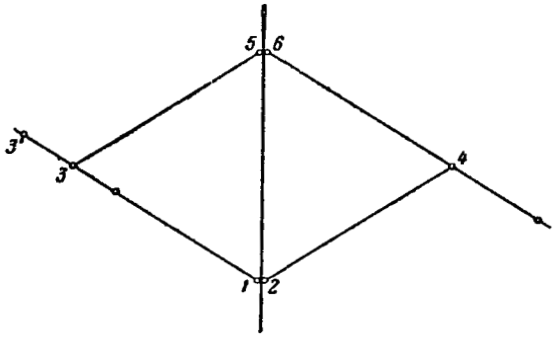

Figure 56

point of the new characteristic, point 2 in the first group is included in the second group of standard elements. Point 1 is located in the main sequence; its address is known - it coincides with the selection address, and therefore it can be immediately transferred to the operational elements of the program.

The CD program must compute points 5 and 6 and perform the requisite preparation for calling the subsequent program. The computations proceed according to the first of the methods presented in Section 10 (see Figure 22). First of all, the coordinates r, t of points 5 and 6 are found. In order to do this, it is sufficient to know points 2 and 4. It is then necessary to determine point 3 at the old characteristic, whose α-characteristic falls at point 5. In the first approximation, it may be assumed that $\alpha_3 = \alpha_1$. Thus, drawing the line $dr = \alpha_1 dt$ from point 5 (its coordinates are already known) and determining its intersection with the line $dr = \beta_1 dt$, which approximately coincides with the old characteristic, we obtain the coordinate t of point 3.

By "examining" the old characteristic, one can find three points on it which are closest to point 3; the remaining quantities at point 3 will be interpolated over them. These three points must be transferred to the special working elements. The preparation for computing points 5 and 6 has now been concluded, and they can be calculated completely according to the computational formulas in Section 10.

After this, points 4 and 6 are stored in the main sequence (at point 6 the sign of r changes); the recording address decreases by 12 units. Point /140 5 is transferred to the second group of standard elements. Point 3' which directly follows behind point 3 is sought on the old characteristic; its address is recorded at the location of the selection address. As may be readily seen, it is now possible to call program $\alpha\beta$, which continues the calculation of the following section of the characteristic in the customary manner.

Program for computing shock wave point (SW program). Similarly to the CD

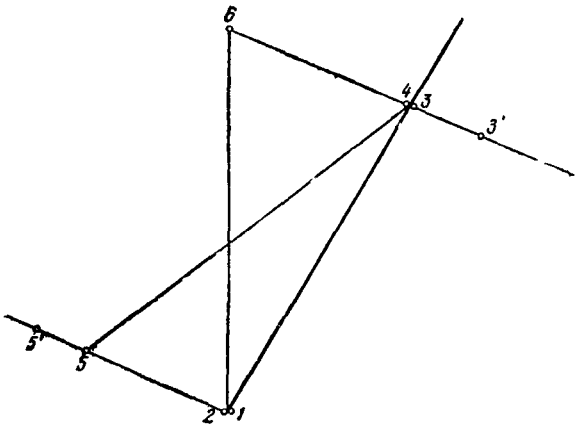

Figure 57

program, this program operates after the αβ program, and calls program αβ after it has finished (but, in contrast to CD, always the same one). When the operation is initiated, point 6 (Figure 57) is the next calculated point of the new characteristic; this point will be "truncated" by the shock wave. As always, it is located in the second group of standard elements. Point 1 is located in the first group of standard elements; the address of point 2 coincides with the address of the selection point.

Knowing points 1 and 2, we may compute the velocity of the shock wave front; knowing point 6, we may compute the slope of this β-characteristic. We may now readily find the coordinates of points 3 and 4, after which point 3 may be determined by interpolation over the section just computed of the new characteristic (which is completed by point 6). Point 5 may be found for the old characteristic in exactly the same way as point 3 for the CD program (Figure 56). The computation of point 4 does not entail any fundamental difficulties after this.

Point 3' which directly follows point 3 is sought on the new characteristic. Point 3 is stored in the main sequence (with negative r) following 3'; the recording address is correspondingly established. Point 5' which /141 directly follows point 5 is found on the new characteristic; its address is assumed to be the selection address. Point 4 is transferred to the second group of standard elements, and the subsequent program is called.

We shall not discuss the operation of the other computational programs (the most important of which are the programs for computing the points to the left and to the right of the boundaries), since their logical structure may be readily formulated by analogy with the programs discussed above. We would only

like to point out certain special cases, which may arise during the operation of the computational programs.

The CD program interpolates point 3 (Figure 56) along the segment of the old β-characteristic. In order to do this, it finds three consecutive points which are closest to point 3 on this segment. However, it is impossible to find three such points. In the first place, three points may not exist on the old characteristic in general - i.e., it may be too short. In addition, it is possible that discontinuity points may be encountered among the points selected for interpolation. This situation arises, for example, when a shock wave approaches a contact discontinuity. Similar situations may exist during the operation of the SW program. The intersection of similar characteristics, etc., may occur during the operation of the αβ program.

All of these cases indicate that the computational program cannot compute the subsequent point, and therefore the computation can be continued only if the length of the β-characteristics is appropriately reduced, or the computation must be stopped, in general. In all such cases, the computational programs effectuate a stop with the appropriate signal. After the start, they call one of the logical programs which determines the sequence of the subsequent calculation. On the one hand, this conforms with the present situation and, on the other hand, with the information introduced previously into the control elements and reflecting the computer requirements regarding the computational sequence in a certain case. When programs are compiled, all of the possible special cases, as well as all of the possible computational variations when they are effectuated, must be stipulated very carefully.

4. CONTROLLING THE SEQUENCE IN WHICH PROGRAMS ARE CALLED

From this point on, we shall employ the term region to designate the portion of the r, t-plane included between two contact discontinuities. The computation is performed by the αβ - and SW programs within each region. The CD programs operate on the boundaries of adjacent regions. The boundary conditions may be defined on the right boundary of the right region and the left boundary of the left region. These boundaries must be computed by special programs, which we shall designate by RB (right boundary) and LB (left boundary).

Let us assume that we must compute a certain β-characteristic. /142
Generally speaking, it intersects several contact discontinuities and shock waves, so that its structure may be expressed (approximately) as is shown in Figure 58. This structure may be coded in the following way, for example. Let us designate all the contact discontinuities by the index 1, and the shock waves by the index 0. We shall use the index 1 for the last (upper) point of the characteristic; we shall not use an index for the lower point. If we now move along this characteristic from the bottom upward and if we record all of the indices encountered in this procedure, we obtain a certain binary code. This code will be as follows for the characteristic shown in Figure 58:

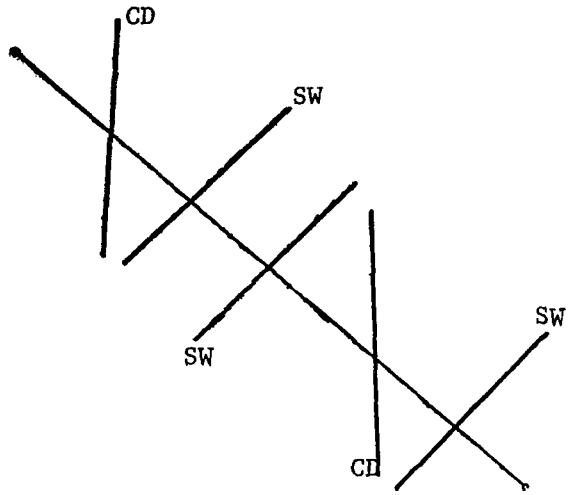

Figure 58

010011

From the number of control elements, let us select a certain element σ and let us write this code in it, beginning with the 0th digit. We obtain the scale of discontinuities reflecting the over-all structure of the regular β-characteristic.

Let us select one element, and let us write one in it in the 0th digit - and zeros in all the remaining places - before the computation is initiated. Let each of the SW and CD programs move this one one digit to the right after they are called. As may be readily seen, when any segment of the characteristic is computed this mobile one will indicate the digit of the element σ which designates the next closest discontinuity. Thus, the αβ programs are able to determine what program must be called - the SW or the CD program.

However, this information is insufficient for an accurate determina- /143 tion of the program to be called. The substances in different regions may satisfy different equations of state, and therefore - for example - the SW programs may differ considerably for different regions. The αβ-programs, which are called by the SW and CD programs, may also be different in different regions. The necessity thus arises of knowing at each moment in which region a given program operates at a given time.

The simplest procedure is to number all of the regions from right to left with the numbers 0, 1, 2,..., and to place a counter in one of the control

elements; each CD program adds one to this counter. Then each computational program may accurately determine which program belongs to it after it is called.

Logical programs participate in the cycle, in addition to computational programs. There are three such programs - B1, B2 and BP, and their function is as follows. They are called when the calculation of each characteristic is terminated, before the beginning of the subsequent computation.

We may now provide a complete description of all the apparatus controlling the sequence in which the programs are called. In order to formulate a command for reading a certain program from the magnetic drum, it is necessary to know two numbers: for example, the number of the first drum element occupied by the program, and the total number of codes for this program. Let us call both of these two numbers the parameters of the given program. Two groups of 12 binary digits are sufficient for recording them - two addresses (for example, of the second and third) of the memory element.

Let us assume that all programs are being computed for problems consisting of no more than three regions. It will be seen that this number may be computed without any particular difficulty. Let us select 13 consecutive (control) elements, and let us number them with x, $x + 1$,....., $x + 12$. Let us place the parameters of all the participating programs in these elements in the following order:

x	$\alpha\beta$	0 region
$x+1$	SW	
$x+2$	CD	
$x+3$	$\alpha\beta$	1 region
$x+4$	SW	
$x+5$	CD	
$x+6$	$\alpha\beta$	2 region
$x+7$	SW	
$x+8$	RB	
$x+9$	LB	
$x+10$	B1	
$x+11$	B2	
$x+12$	BP	

These parameters occupy only 24 digits of each element; the remaining /144 digits may be utilized for other information (the selection and recording addresses may be placed there, for example). In addition, three numbers are stored in the control elements - the number of the first region (where the lower point of the characteristic is located), the number of the last region (where the upper point is located), and the current number of the region. The current number of the region coincides with the number of the first region before the initial computation of the regular characteristic. After this, each CD program adds one to it, so that at the end of the computation it must coincide with the number of the last region (if a special case does not occur, due

to which the given characteristic cannot be computed until the end).

The last (upper) point of the next β-characteristic can either belong or not belong to the left boundary. Thus, in the problem shown in Figure 54, the first case occurs at the beginning of the computation; the second case occurs at the end of the computation. In the first case, at the end of the computation the characteristics of the αβ-program must call the LB program; in the second case, the B1 program is called directly (the LB program always calls B1). Consequently, we must thus be able to distinguish between these two cases. One digit in one of the control elements can be assigned, in order to do this. The presence of one in this digit will signify the presence of the left boundary; zero in this digit will indicate its absence.

Let us assume that the αβ-program is in operation at any moment. Let us examine in detail the manner in which it produces the call of the subsequent program.

First of all, the program compares the number of the last region with the current number of the region. Two cases are thus possible:

1. Number does not coincide. The program determines the contents of the element digit σ indicated by the mobile one (see above). Let these contents be τ (τ equals zero or one). Let n be the current number of the region. The program selects the parameters from the element with the number

$$\varkappa + 3n + \tau + 1.$$

As may be readily seen, this will be the parameters of the SW or CD program corresponding to the given region.

2. Numbers coincide. Just as previously, the program finds the quantity τ. If τ = 0, then the parameters are selected from the element with the number ϰ + 3n + 1 (the SW program). If τ = 1, the presence of the left boundary is tested, and - depending on its condition - either the parameters LB or B1 are selected.

The SW and RB programs select the parameters from the element with the number ϰ + 3n (the αβ-program in the given region). The CD programs proceed in the same manner, only it takes place after a change in the current number of the region. The LB program always calls B1, and the latter - B2. This sequence of calls is disturbed in special cases (see above); then any computational program calls the program B1.

5. LOGICAL PROGRAMS /145

The left logical program B1 operates immediately after the regular characteristic has been computed. The basic functions of this program consist of

processing the upper end of the calculated characteristic, changing the computational regime in special cases (i.e., when this program is called "outside of the sequence"), and producing so-called small outputs.

The region to be investigated may be limited above either by a certain α-characteristic, or by certain cross section $t = t_0$. The quantity t_0 must be given, i.e., it is included among constants which determine the problem as a whole. In the latter case, the upper ends of the β-characteristics must be "truncated" from time to time, or - as may be readily seen - the upper point of the new characteristic will always lie above the upper point of the old one. "Truncation" of the characteristic may be reduced to the following operations:

(a) determination of the last point remaining on the characteristic. In the case under consideration, one-two points must remain above the cross section $t = t_0$ in order that interpolation may be subsequently performed over $t = t_0$ and the cross section point may be determined.

(b) the recording at this point of the end of the characteristic, i.e., the minus sign for r;

(c) a change in the scale σ, or discontinuity points may occur among the points which are discarded;

(d) a change in the number of the last region, if there is a contact discontinuity point among the points which are discarded.

A change in the computational regime in special cases naturally depends on the nature of the case. As a rule, it may be primarily reduced to "truncating" part of the β-characteristic. Thus, if the shock wave approaches the contact discontinuity, so that one of the SW and CD programs discovers a shortage of points for the interpolation (see above), it is then necessary to "truncate" part of the β-characteristic line above this contact discontinuity.

The "small outputs" produced also by the B1 program primarily include the output of the upper and lower points of the computed characteristic, as well as the points (binary) of all the discontinuities. All of these points may be determined with no difficulty, since they have the appropriate designations. In addition it is possible to determine the points of the given cross sections (t = const) and the trajectory (R = const). Interpolation over the characteristic is required here. The B1 program contains the corresponding block.

Let us describe the operation of the second logical program - B2. If the basic problem of the B1 program is "to process" the computed β-characteristic, then the B2 program is concerned with the preparation for computing the new characteristic. We must point out that, in essence, both of these programs should be regarded as one "current" program. Their separation into two /146 programs is dictated by purely technical considerations (economy of the operative memory).

The sequence of points comprising the line of initial data is stored in the memory of the machine, along with the main sequence (the β-characteristic). Generally speaking, this line consists of several fragments adjacent to each other; each of these fragments is either a spacelike line (for example, the t = const line) or a fragment of the α-characteristic. The transition points from one segment to another (angular points) are in a certain sense special points for the line of initial data. In addition, the line of initial data can contain discontinuity points - contact discontinuities and shock waves.

All of these points must have the appropriate designations, in order that the program may recognize them. In order to do this, it is possible to employ (in a more comprehensive form) the same method used to record the discontinuity points in the main sequence. The quantities r and R may always be assumed to be positive. In addition, they are stored in the memory of the machine, as always, in the form of normalized binary numbers, so that the first digit of the mantissa must always be one for them. Thus, both for r and R the contents of the two digits - the sign of the number and of the first digit of the mantissa - was known beforehand. Therefore, these digits may arbitrarily change when they are stored in the sequence (after a point is transferred to the operational elements, in order to perform the computation it is necessary to restore the normal form of these digits each time). This makes it possible to assign 16 different designations to the points.

The preparation for computing the regular characteristic encompasses the following two steps at least:

(a) the transfer of the point of initial data to the second group of standard elements;

(b) establishment of the initial state of the selection and recording addresses.

We should point out that the initial state of the selection and recording addresses depends on the nature of the initial data segment on which the next point lies. If this segment is spacelike, then the selection and recording addresses must coincide. If the initial data segment is a segment of the α-characteristic, the recording address must be six units greater than the selection address (the last point of the old characteristic does not participate in the computation).

When the next point of initial data is a discontinuity point, additional operations are generally required, the most important of which are:

(c) a change in the scale of the discontinuity σ (in view of the formation of a new discontinuity);

(d) a change in the number of the first region (when this new discontinuity is a contact discontinuity).

The last - the third - logical program (BP) produces the controlled repetitions of the computation and produces the so-called large outputs. /147 As always, the operation of the machine must be controlled. For this purpose, the computation is divided into fragments (let us say there are 10 characteristics per fragment); the computation of each fragment is repeated. The results are compared and, when they coincide, the state of the operative memory is recorded on magnetic tape, and the computation is continued. Since this procedure is well known to every experienced programmer, we shall not describe it in detail.

Immediately after the results of the next fragment are recorded on the tape, the BP program produces the output of the main sequence on the punch ("large output").

In addition to the computational and logical programs enumerated above, one "actuating" program participates in the operation. Its assignment is to introduce the initial data, to convert them into a binary system, to arrange them in the memory at the requisite places, to assign the appropriate designations to the special points, to formulate the control elements, etc. This program operates once, before the initiation of the computation.

In very general outlines, these are the principles underlying the programs for performing computations according to the characteristics method. We should point out once more that the solution presented above for different computational and logical problems may in no way be regarded as the only possible solution.

APPENDIX 2. EXAMPLE

In conclusion, we would like to present one specific example illustrating the manner in which a gasodynamics problem is computed according to the method presented above.

The structure of the problem is as follows. At the initial moment, there is a spherical volume having the radius r = 1 filled with an ideal gas having an adiabatic index $\varkappa = \frac{5}{3}$ with the density $\rho_0 = 1$ and pressure $p_0 = 20$. This volume is enclosed by a layer of the same gas having the same density, but with zero pressure (it may be assumed that these substances are initially divided by a solid, infinitely thin, spherical wall). The outer radius of this layer is 2. Finally, on the outside there is a spherical envelope made of a substance having an equation of state such as (5.15) - (5.18) with an outer radius of 2.5, which is characterized by the following parameters: $\varkappa = 3$, $\rho_0 = 5$, $c_0 = 3$, $p_0 = 0$. This entire system is surrounded by a vacuum.

At the time t = 0, let us assume that the partition enclosing the inner spherical volume of the substance with the pressure p = 20 suddenly is removed. As it expands, the substance sets in motion the layers surrounding it; the motion will be spherically symmetrical.

Figure 59 presents the over-all picture of the motion on the r, t-plane, obtained by employing the characteristics method for the computation. After the decay of the initial discontinuity (point 1), the boundary of the inner substance begins to move to the right (line 1 - 2); a shock wave (line 1 - 3) arises in the middle layer before it. An expansion wave defined by the characteristics 1 - 4 and 1 - 5 appears in the inner volume. A new shock wave 5 - 6, which results from the intersection of the β-characteristics, arises at point 5.

The shock wave 1 - 3, reaching the boundary with the outer layer, is reflected from it (line 3 - 2); a shock wave 3 - 7 passes into the outermost layer. The reflected wave 3 - 2, reaching boundary 1 - 2, decays into a shock wave 2 - 8 and an expansion wave which is defined by the characteristics 2 - 9 and 2 - 10. The boundary of the inner layer continues the motion along the line 2 - 11.

The shock wave 3 - 7, reaching the free boundary (R = 2.5), is discharged by the expansion wave included between the lines 7 - 12 and 7 - 13. The last characteristic (shown in the figure by the dashed line) is that lying in the region of negative pressure; in other words, a separation develops in /149 the outer envelope. The origin is taken on the line 7 - 14, where the pressure p vanishes. This line becomes timelike for point 14, so that a wave encompassing the separation (14 - 15) is initiated at point 14. The separation region is crosshatched in the figure.

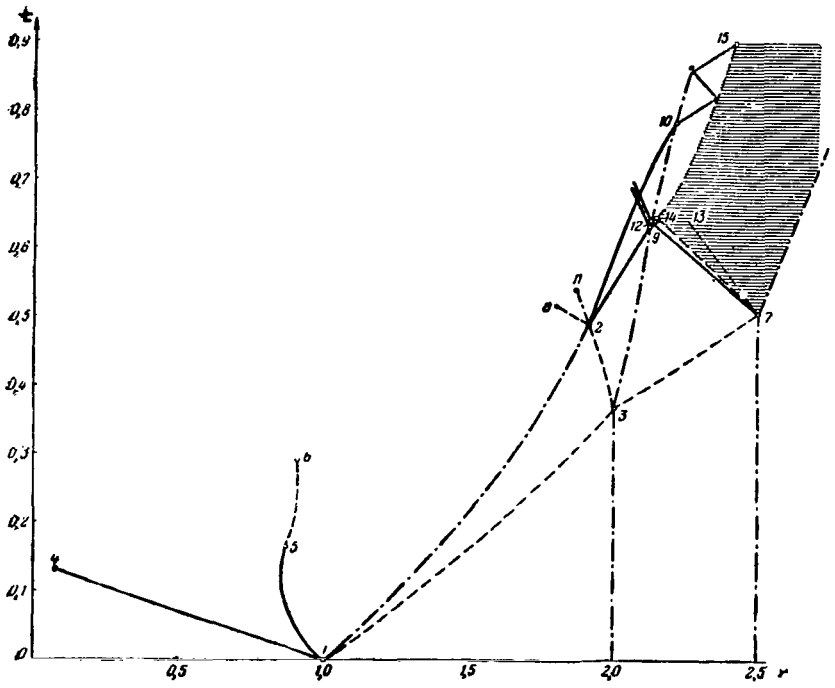

Figure 59

The problem was computed on an electronic "Strela" computer according to the program described in Appendix 1. The characteristic 1 - 4 served as the initial data line; on this characteristic, u, ρ, p retain constant values ($u = 0$, $\rho = 1$, $p = 20$) as may be readily seen. The computation was divided into several sections; the separation was computed by hand.

All of the computational elements were adequately discussed in the preceeding sections, and therefore we shall not present them.

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